Control Systems

Definitions

1- **Systems**: A system is a combination of components that act together and perform a certain objective.

2. **Reference Input**: It is the actual signal input to the control system.

3. **output (controlled variable)**: The quantity that must be maintained at a prescribed value.

4. **Open-Loop Control System**: A system in which the output has no effect upon the input signal.

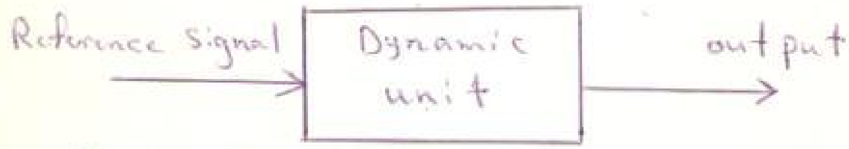


Fig. Open-Loop Control System.

5. **Closed-Loop Control system**: A system in which the output has an effect upon the input quantity in such a manner as to maintain the desired output value.

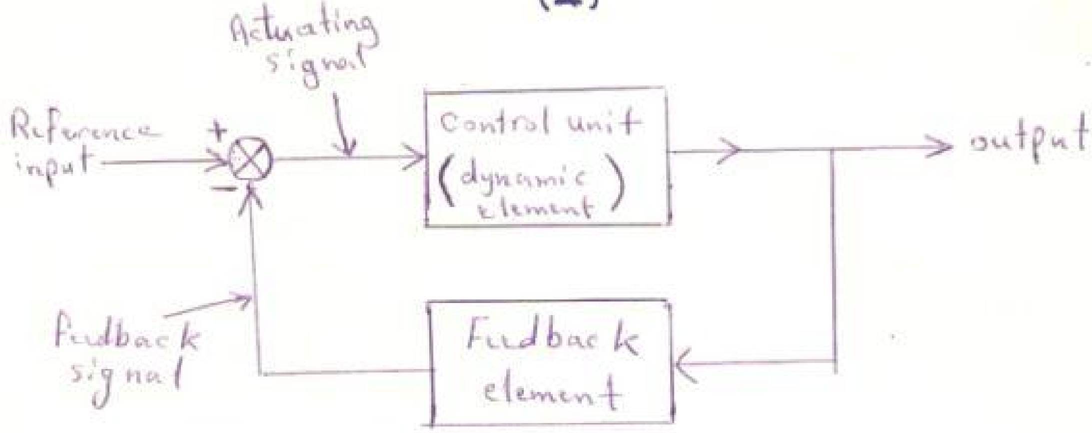


Fig. Closed-Loop Control System

6. plants : A plant may be a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation.

7. Control unit (dynamic element) : The unit that reacts to an actuating signal to produce a desired output. This unit does the work of Controlling the output and thus may be a power amplifier.

8. Feedback Element : The unit that provides the means for feeding back the output quantity, or a function of the output, in order to Compare it with the reference input.

9. Actuating Signal : The signal that is the difference between the reference input and the feedback signal. It actuates the control unit in order to maintain the output at the desired value.

Transfer function

The transfer function of a Linear time-invariant differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial Conditions are zero.

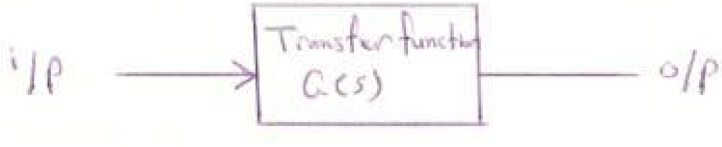
$$a_0 y^{n} + a_1 y^{n-1} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 x^{n} + b_1 x^{n-1} + \cdots$$
$$+ b_{m-1}\dot{x} + b_m x \qquad (n \geqslant m)$$

Where y ---- output of the system
x ---- input of the system

$$\text{transfer function} = G(s) = \left.\frac{\mathcal{L}\,\text{output}}{\mathcal{L}\,\text{input}}\right|_{\text{Zero initial conditions}}$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^{m} + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^{n} + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Block diagram: A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

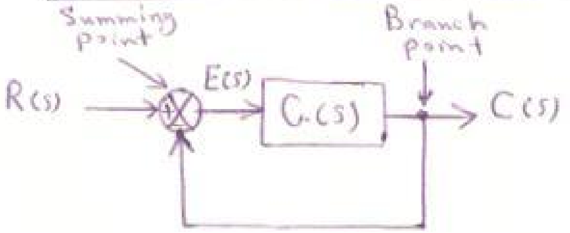


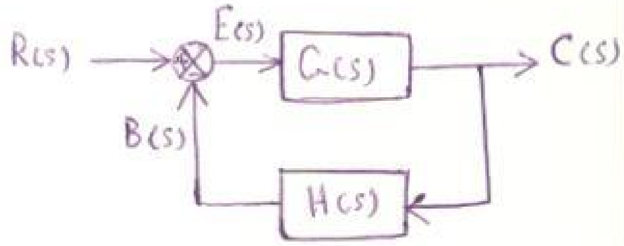
Summing point: A circle with a cross is the symbol that indicates a summing operation.

Fig. Summing point

Branch point: A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block diagram of a closed Loop System:





* open-Loop transfer function $= \dfrac{B(s)}{E(s)} = G(s)\,H(s)$

* feed forward transfer function $= \dfrac{C(s)}{E(s)} = G(s)$

* closed-Loop transfer function:

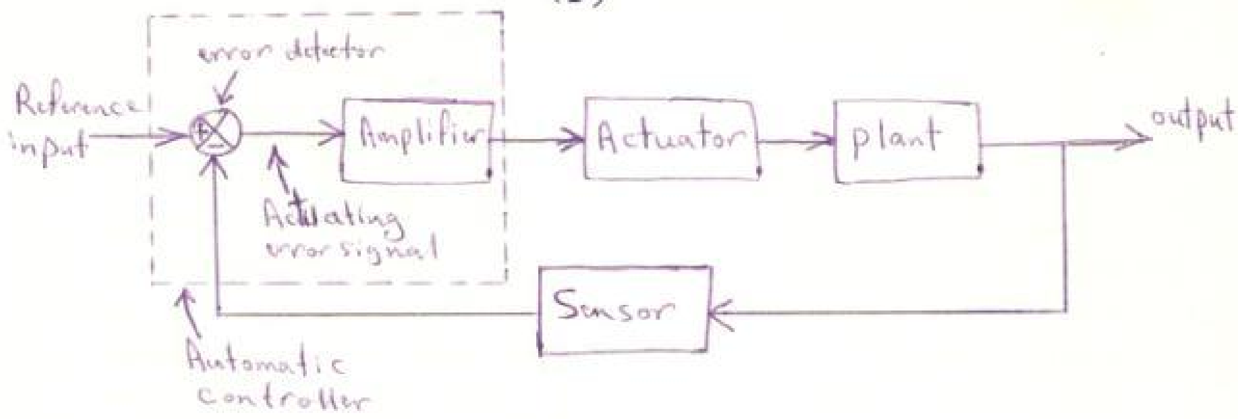
$$C(s) = G(s)\,E(s)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)\,C(s)$$

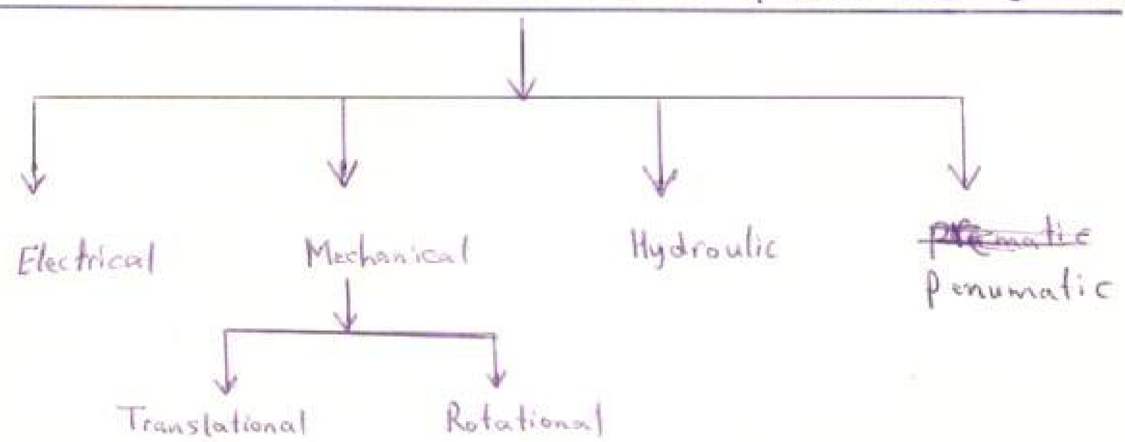
$$C(s) = G(s)\left[R(s) - H(s)\,C(s)\right]$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$ = closed-Loop transfer function

Automatic Controllers: An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value.



Mathmatical Representation of Control Components and Systems;



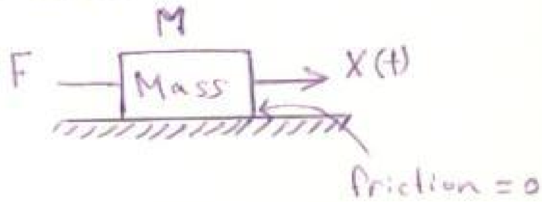
①. Mechanical Components;

a. Translational Mechanical Motion.
b. Rotational Mechanical Motion.

(a) Transtational Mechanical Motion;

(1) F = Mass $\times$ acceleration

$$= M \cdot \frac{d^2x}{dt^2}$$



$$F(t) = M \frac{d^2X(t)}{dt^2}$$

$X(t)$ = Displacement at any time (t).

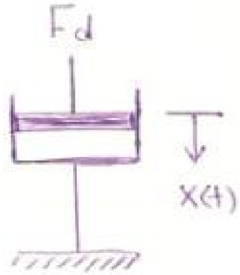
$$F_m(s) = M\,s^2\,X(s)$$

$$\frac{X(s)}{F_m(s)} = \frac{1}{Ms^2} = \frac{1}{MD^2}$$

where D-operator $= \frac{d}{dt}$

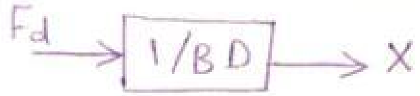
$D = S$ under zero I.C's

(2) $$F_d(t) = B\,\frac{dx(t)}{dt}$$



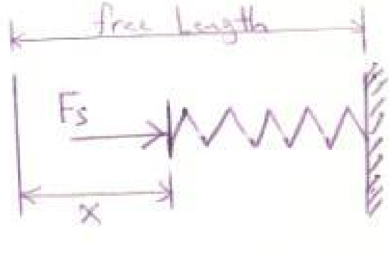
$$F_d(s) = B\,S\,X(s) = B\,D\,X(s)$$

$$\frac{X(s)}{F_d(s)} = \frac{1}{BD}$$



where B.... Damping factor or coefficient of the damper

(3) $$F_s = K\,X$$

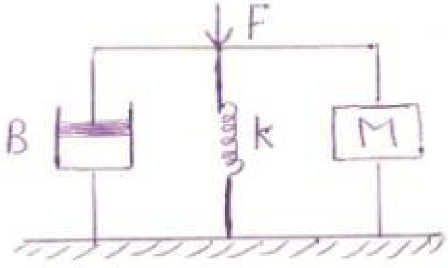


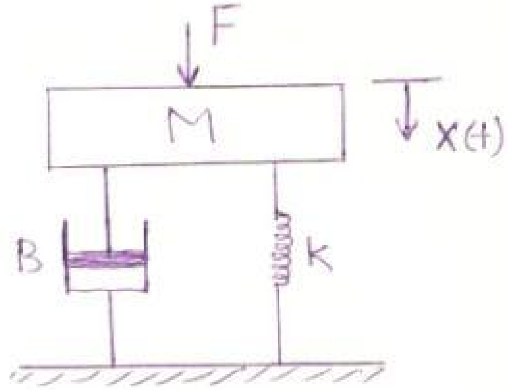
where K.... elastance of spring

$$F_s(s) = K\,X(s)$$

$$\frac{X(s)}{F_s(s)} = 1/K$$

* Series Mechanical Combination :-





$E = Z_T \, I$ (in electrical circuit)

$$F = Z_T X$$

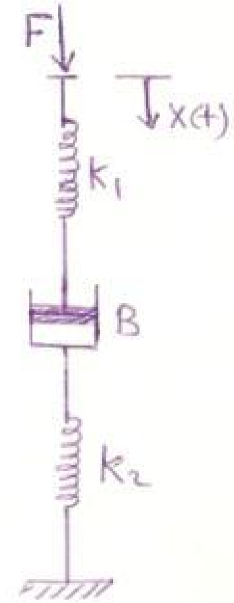
$$Z_T = Z_m + Z_d + Z_s = MS^2 + BS + K$$

$$F(s) = (MS^2 + Bs + k)\, X(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{1}{MS^2 + Bs + K}$$

$$F(s) \longrightarrow \boxed{1/MS^2+BS+K} \longrightarrow X(s)$$

* Parallel Mechanical Combination :-



$$\frac{1}{Z_T} = \frac{1}{Z_{s_1}} + \frac{1}{Z_d} + \frac{1}{Z_{s_2}}$$

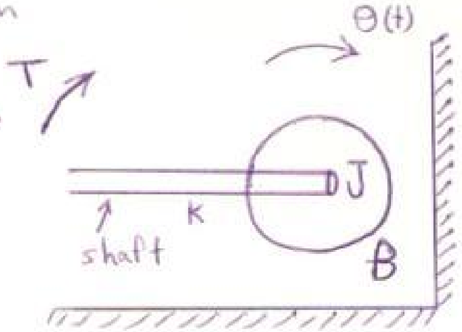
$$= \frac{1}{K_1} + \frac{1}{BS} + \frac{1}{K_2}$$

$$F \longrightarrow \boxed{\frac{1}{Z_T}} \longrightarrow X$$

(b) Rotational Mechanical Motion :

Net Torque = change of Momentum

= moment of inertia * Angular Acceleration



$$T - T_d - T_s = J \frac{d^2\theta(t)}{dt^2}$$

$$T - B\frac{d\theta(t)}{dt} - K\theta(t) = J\frac{d^2\theta(t)}{dt^2}$$

$$T(s) = (Js^2 + Bs + K)\,\theta(s)$$

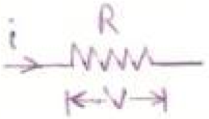
$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

* $T_s = 0$ for riggid shaft

$$\therefore \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs} \quad \text{for riggid shaft}$$

$$\frac{\dot{\theta}(s)}{T(s)} = \frac{1}{Js + B}$$

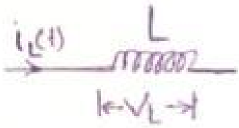
② - Electrical Components:



$$V = iR$$

$$V(s) = I(s)\,R$$

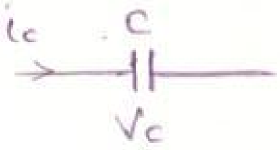
$$\therefore Z_R(s) = R$$



$$V_L = L\frac{di_L(t)}{dt}$$

$$V_L(S) = LS\,I(S)$$

$$Z_L(S) = LS$$



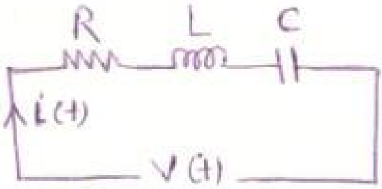
$$V_c = \frac{1}{c}\int i_c\,dt$$

$$\frac{dV_c}{dt} = \frac{i_c}{C}$$

$$S V_c(S) = \frac{I_c(S)}{C}$$

$$V_c(S) = \frac{1}{CS}\,I(S)$$

$$\therefore Z_c(S) = \frac{1}{CS}$$



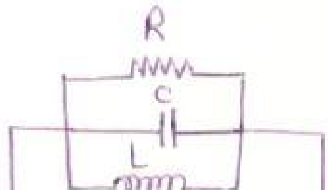
$$Z_T(s) = R + LS + \frac{1}{CS}$$

$$V(t) = Z_T\, i(t)$$

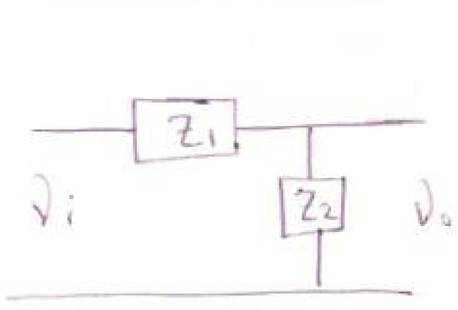
$$V(s) = Z_T(s)\,I(s) = I(s)\left(R + LS + \frac{1}{CS}\right)$$

$$\frac{I(s)}{V(s)} = \frac{1}{R + LS + \frac{1}{CS}}$$

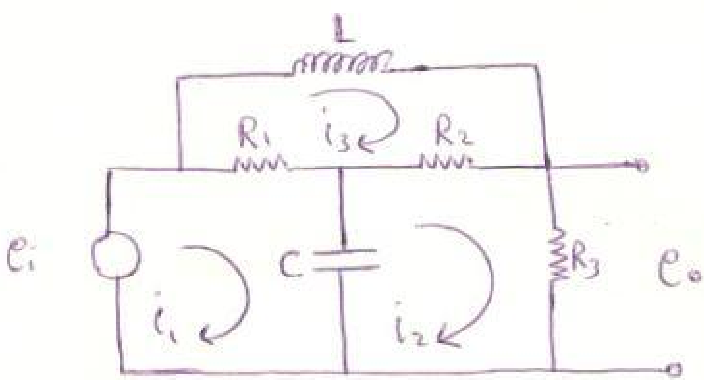
$$\frac{1}{Z_T} = \frac{1}{R} + \frac{1}{LS} + CS$$



$$\frac{v_o}{v_i} = \frac{Z_2}{Z_2 + Z_1}$$



Ex.



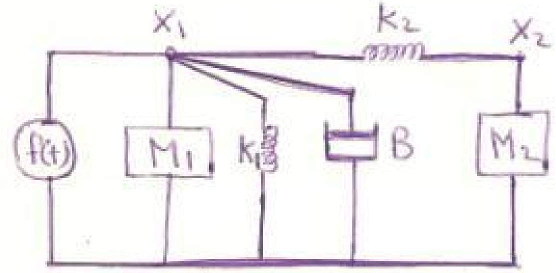
$$e_i(t) = R_1(i_1 - i_3) + \frac{1}{C}\int (i_1 - i_2)\,dt \quad \text{—— (1)}$$

$$0 = \frac{1}{C}\int (i_2 - i_1)\,dt + R_2(i_2 - i_3) + i_3 R_3 \quad \text{—— (2)}$$

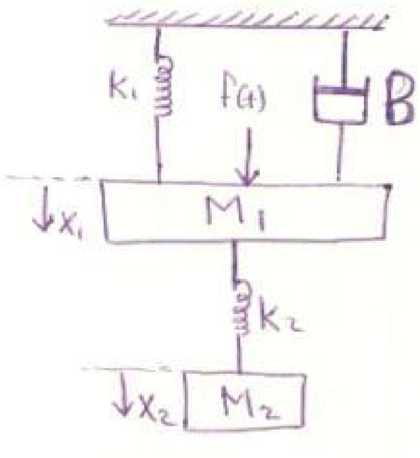
$$0 = L\frac{di_3}{dt} + R_1(i_3 - i_1) + R_2(i_3 - i_2) \quad \text{—— (3)}$$

$$e_o = i_2 R_3 \quad \text{—— (4)}$$

Ex.



equivalent circuit



Node x_1 :

$$f(t) = M_1 \ddot{x}_1 + B\dot{x}_1 + K_1 x_1 + K_2(x_1 - x_2) \quad \text{—— (1)}$$

Node x_2 :

$$0 = M_2 \ddot{x}_2 + K_2(x_2 - x_1) \quad \text{—— (2)}$$

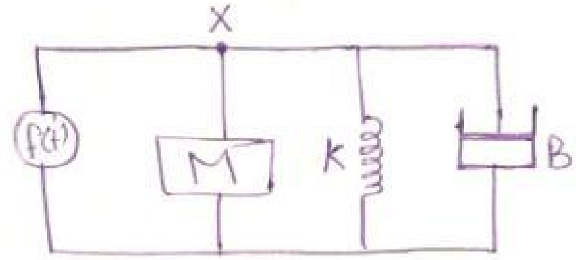
Taking Laplace Transfer

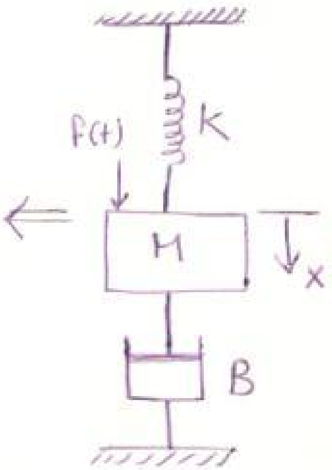
$$F(s) = (M_1 s^2 + Bs + K_1 + K_2) X_1(s) - K_2 X_2(s) \quad \text{—— ①}$$

$$0 = (M_2 s^2 + K_2) X_2(s) - K_2 X_1(s) \quad \text{—— ②}$$

Ex:

$\frac{X(s)}{F(s)}$?



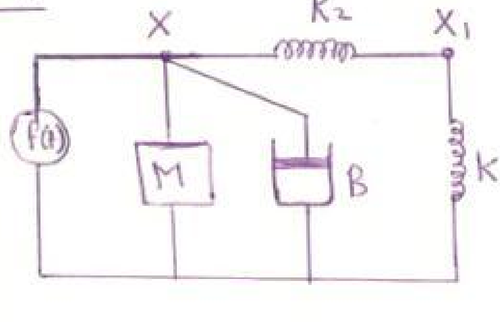


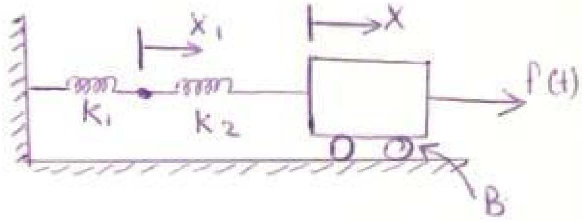
$$f(t) = M\ddot{x} + B\dot{x} + Kx$$

$$F(s) = (Ms^2 + Bs + K)X(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

Ex:





Node X:

$$f(t) = M\ddot{x} + B\dot{x} + K_2(x - x_1) \quad \text{—— ①}$$

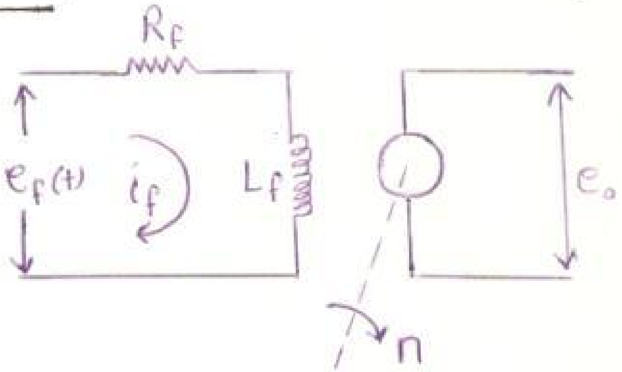
$$F(s) = (Ms^2 + Bs + K_2)X(s) - K_2 X_1(s)$$

Node X_1:

$$0 = K_1 X_1 + K_2(X_1 - X) \quad \text{—— ②}$$

$$0 = (K_1 + K_2) X_1(s) - K_2 X(s)$$

T.F for a D.C generator:



$$\frac{E_g(s)}{E_f(s)} = ?$$

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

$$E_f(s) = (R_f + L_f S)\, I_f(s) \quad \text{—— (1)}$$

$e_g \propto n\Phi$ where Φ -- flux density

$\Phi \propto i_f \Rightarrow \Phi = k_2 i_f$

$e_g = k_1 n k_2 i_f$

$e_g = k_g i_f$ (k_g = generator constant)

$$E_g(s) = k_g I_f(s) \quad \text{—— (2)}$$

$$\frac{E_g(s)}{E_f(s)} = \frac{k_g}{R_f + L_f S} = \frac{k}{1+TS}$$

where

$T = \frac{L_f}{R_f}$ = electric time constant

$K = \frac{k_g}{R_f}$

T.F of Tachometer generator:

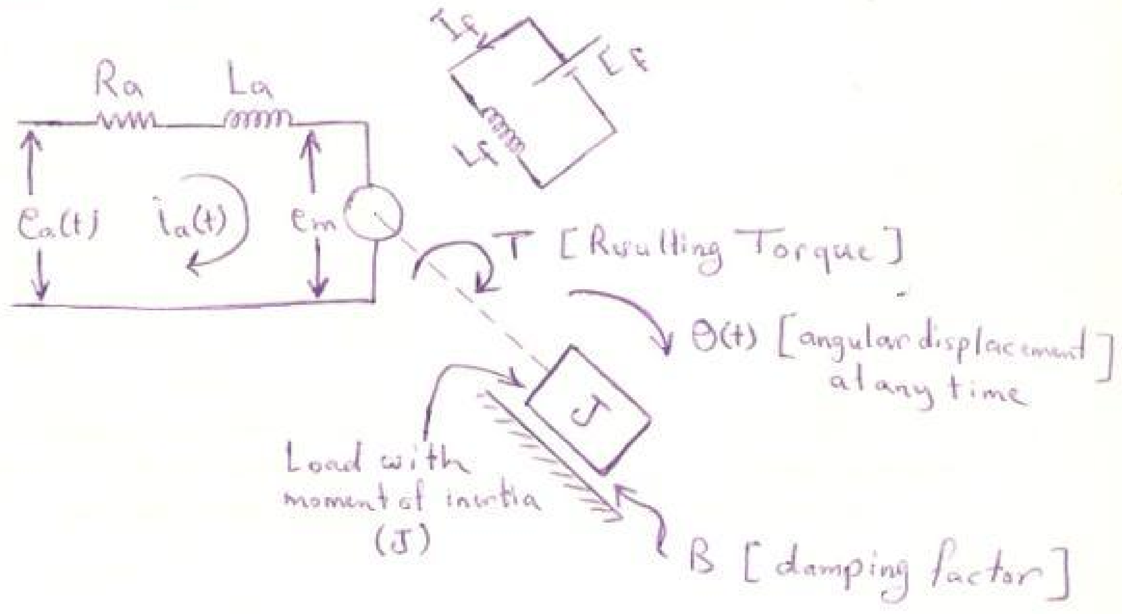
It is a tacho generator. It is a d.c. generator with a permanent magnetic, therefore, compared with d.c. generator we find that:

$e_g \propto n\Phi$

where n = speed of rotating
Φ = mag. flux density

In the case of tacho generator we have a parmanent magnetic, therefore, the flux density is constant. Hence the (e_g) becomes $e_g = K_T n$, where K_T is tacho constant.

T.F for Armature Control D.C Servo motor (ACDSM):



e_m -- is the induced e.m.f

$$\frac{\theta(s)}{E_a(s)} = ?$$

$$e_a(t) = i_a R_a + L_a \frac{di_a(t)}{dt} + e_m \quad \text{—— ①}$$

$$E_a(s) - E_m(s) = (R_a + L_a S)\, I_a(s) \quad \text{—— ①}$$

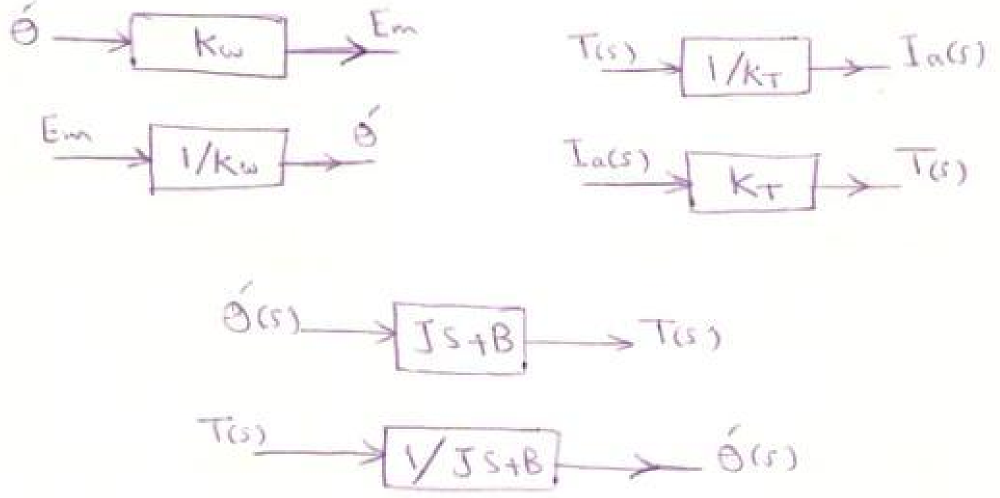
$$e_m = k_\omega \theta'$$

$$E_m(s) = K_\omega \theta'(s) = K_\omega S\, \theta(s) \quad \text{—— ②}$$

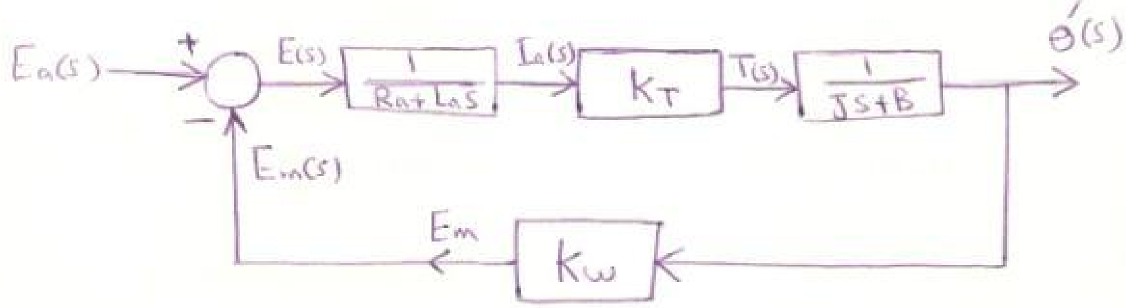
$$T \propto i_a(t) \Rightarrow T(s) = k_T\, I_a(s) \quad \text{—— ③}$$

$$T(t) = J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt}$$

$$T(s) = (JS^2 + BS)\,\theta(s) = (JS + B)\,\theta'(s) \quad \text{—— ④}$$



Closed Loop:



8

State-space methods for control system design

8.1 The state-space-approach

The classical control system design techniques discussed in Chapters 5–7 are generally only applicable to

(a) Single Input, Single Output (SISO) systems
(b) Systems that are linear (or can be linearized) and are time invariant (have parameters that do not vary with time).

The state-space approach is a generalized time-domain method for modelling, analysing and designing a wide range of control systems and is particularly well suited to digital computational techniques. The approach can deal with

(a) Multiple Input, Multiple Output (MIMO) systems, or multivariable systems
(b) Non-linear and time-variant systems
(c) Alternative controller design approaches.

8.1.1 The concept of state

The state of a system may be defined as: 'The set of variables (called the state variables) which at some initial time t_0, together with the input variables completely determine the behaviour of the system for time $t \geq t_0$'.

The state variables are the smallest number of states that are required to describe the dynamic nature of the system, and it is not a necessary constraint that they are measurable. The manner in which the state variables change as a function of time may be thought of as a trajectory in n dimensional space, called the *state-space*. Two-dimensional state-space is sometimes referred to as the *phase-plane* when one state is the derivative of the other.

8.1.2 The state vector differential equation

The state of a system is described by a set of first-order differential equations in terms of the state variables $(x_1, x_2, \ldots, x_n)$ and input variables $(u_1, u_2, \ldots, u_n)$ in the general form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m\end{aligned} \tag{8.1}$$

The equations set (8.1) may be combined in matrix format. This results in the state vector differential equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{8.2}$$

Equation (8.2) is generally called the state equation(s), where lower-case boldface represents vectors and upper-case boldface represents matrices. Thus

x is the n dimensional state vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{8.3}$$

u is the m dimensional input vector

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \tag{8.4}$$

A is the $n \times n$ system matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{8.5}$$

B is the $n \times m$ control matrix

$$\begin{bmatrix} b_{11} & \ldots & b_{1m} \\ b_{21} & \ldots & b_{2m} \\ \vdots & & \\ b_{n1} & \ldots & b_{nm} \end{bmatrix} \tag{8.6}$$

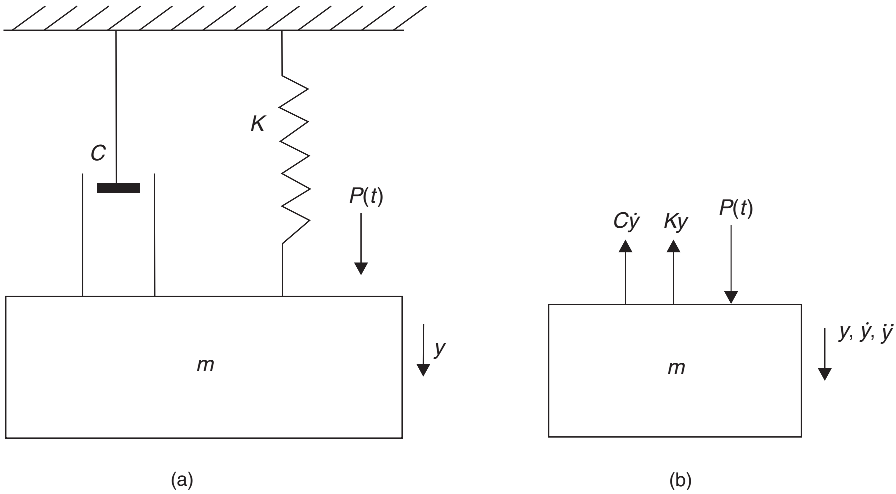


Fig. 8.1 Spring–mass–damper system and free-body diagram.

In general, the outputs $(y_1, y_2, \ldots, y_n)$ of a linear system can be related to the state variables and the input variables

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{8.7}$$

Equation (8.7) is called the output equation(s).

Example 8.1
Write down the state equation and output equation for the spring–mass–damper system shown in Figure 8.1(a).

Solution
State variables

$$x_1 = y \tag{8.8}$$

$$x_2 = \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{x}_1 \tag{8.9}$$

Input variable

$$u = P(t) \tag{8.10}$$

Now

$$\sum F_y = m\ddot{y}$$

From Figure 8.1(b)

$$P(t) - Ky - C\dot{y} = m\ddot{y}$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{K}{m}y - \frac{C}{m}\dot{y} + \frac{1}{m}P(t) \tag{8.11}$$

From equations (8.9), (8.10) and (8.11) the set of first-order differential equations are

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{m}x_1 - \frac{C}{m}x_2 + \frac{1}{m}u\end{aligned} \tag{8.12}$$

and the state equations become

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\dfrac{K}{m} & -\dfrac{C}{m}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \dfrac{1}{m}\end{bmatrix}u \tag{8.13}$$

From equation (8.8) the output equation is

$$y = [1 \quad 0]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \tag{8.14}$$

State variables are not unique, and may be selected to suit the problem being studied.

Example 8.2
For the *RCL* network shown in Figure 8.2, write down the state equations when

(a) the state variables are $v_2(t)$ and $\dot{v}_2$
(b) the state variables are $v_2(t)$ and $i(t)$.

Solution
(a)

$$\begin{aligned}x_1 &= v_2(t) \\ x_2 &= \dot{v}_2 = \dot{x}_1\end{aligned} \tag{8.15}$$

From equation (2.37)

$$LC\frac{d^2v_2}{dt^2} + RC\frac{dv_2}{dt} + v_2 = v_1(t) \tag{8.16}$$

From equations (8.15) and (8.16) the set of first-order differential equations are

$$\dot{x}_1 = x_2$$

$$x_2 = -\frac{1}{LC}x_1 - \frac{RC}{LC}x_2 + \frac{1}{LC}u \tag{8.17}$$

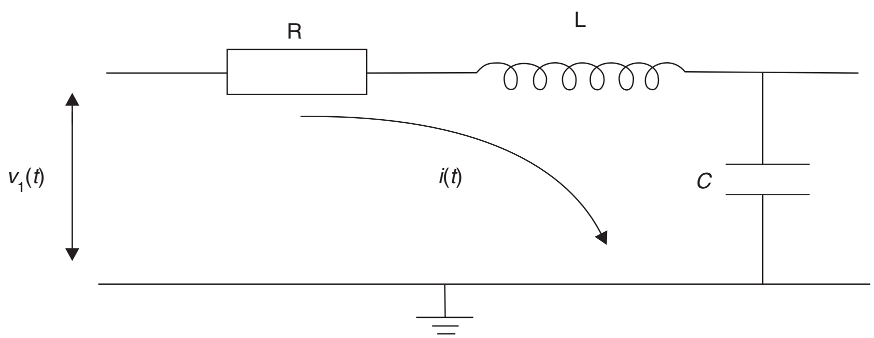


Fig. 8.2 *RCL* network.

and the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & \dfrac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u \tag{8.18}$$

(b)

$$\begin{aligned} x_1 &= v_2(t) \\ x_2 &= i(t) \end{aligned} \tag{8.19}$$

From equations (2.34) and (2.35)

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -v_2(t) - Ri(t) + v_1(t) \tag{8.20}$$

$$C\frac{\mathrm{d}v_2}{\mathrm{d}t} = i(t) \tag{8.21}$$

Equations (8.20) and (8.21) are both first-order differential equations, and can be written in the form

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}x_2 \\ \dot{x}_2 &= -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \end{aligned} \tag{8.22}$$

giving the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} u \tag{8.23}$$

Example 8.3
For the 2 mass system shown in Figure 8.3, find the state and output equation when the state variables are the position and velocity of each mass.

Solution
State variables

$$\begin{aligned} x_1 &= y_1 \quad x_2 = \dot{y}_1 \\ x_3 &= y_2 \quad x_4 = \dot{y}_2 \end{aligned}$$

System outputs

$$y_1, y_2$$

System inputs

$$u = P(t) \tag{8.24}$$

For mass m_1

$$\sum F_y = m_1\ddot{y}_1$$

$$K_2(y_2 - y_1) - K_1y_1 + P(t) - C_1\dot{y}_1 = m_1\ddot{y}_1 \tag{8.25}$$

For mass m_2

$$\sum F_y = m_2\ddot{y}_2$$

$$-K_2(y_2 - y_1) = m_2\ddot{y}_2 \tag{8.26}$$

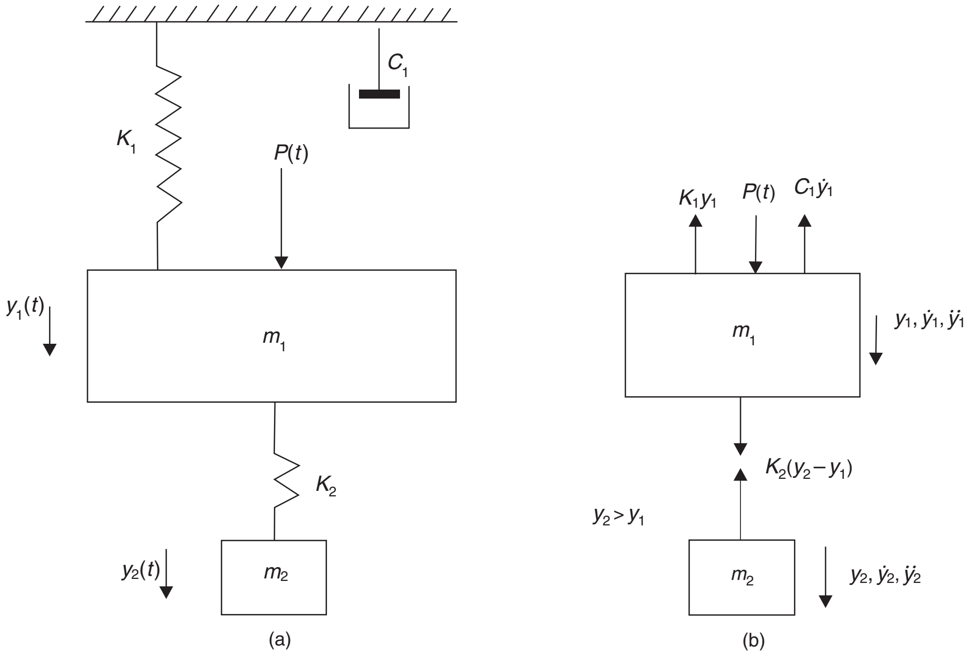


Fig. 8.3 Two-mass system and free-body diagrams.

From (8.24), (8.25) and (8.26), the four first-order differential equations are

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \left(-\frac{K_1}{m_1} - \frac{K_2}{m_1}\right)x_1 - \frac{C_1}{m_1}x_2 + \frac{K_2}{m_1}x_3 + \frac{1}{m_1}u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{K_2}{m_2}x_1 - \frac{K_2}{m_2}x_3
\end{aligned} \tag{8.27}$$

Hence the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\dfrac{K_1+K_2}{m_1}\right) & -\dfrac{C}{m_1} & \dfrac{K_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{m_2} & 0 & -\dfrac{K_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u \tag{8.28}$$

and the output equations are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{8.29}$$

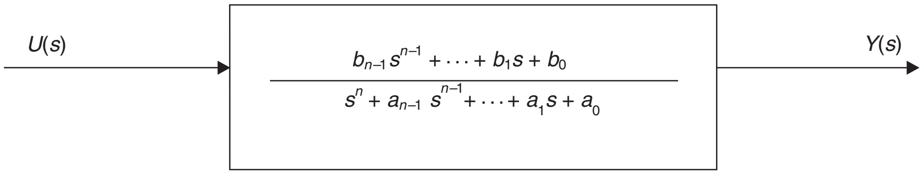


Fig. 8.4 Generalized transfer function.

8.1.3 State equations from transfer functions

Consider the general differential equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_{n-1}\frac{\mathrm{d}^{n-1} u}{\mathrm{d}t^{n-1}} + \cdots + b_1\frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u \tag{8.30}$$

Equation (8.30) can be represented by the transfer function shown in Figure 8.4.

Define a set of state variables such that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots \quad & \quad \vdots \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + u \end{aligned} \tag{8.31}$$

and an output equation

$$y = b_0 x_1 + b_1 x_2 + \cdots + b_{n-1}x_n \tag{8.32}$$

Then the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{8.33}$$

The state-space representation in equation (8.33) is called the controllable canonical form and the output equation is

$$y = [\, b_0 \quad b_1 \quad b_2 \quad \ldots \quad b_{n-1}\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \tag{8.34}$$

Example 8.4 (See also Appendix 1, *examp84.m*)

Find the state and output equations for

$$\frac{Y}{U}(s) = \frac{4}{s^3 + 3s^2 + 6s + 2}$$

Solution
State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{8.35}$$

Output equation

$$y = [4 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.36}$$

Example 8.5
Find the state and output equations for

$$\frac{Y}{U}(s) = \frac{5s^2 + 7s + 4}{s^3 + 3s^2 + 6s + 2}$$

Solution
The state equation is the same as (8.35). The output equation is

$$y = [4 \quad 7 \quad 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.37}$$

8.2 Solution of the state vector differential equation

Consider the first-order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(t) + bu(t) \tag{8.38}$$

where $x(t)$ and $u(t)$ are scalar functions of time. Take Laplace transforms

$$sX(s) - x(0) = aX(s) + bU(s) \tag{8.39}$$

where $x(0)$ is the initial condition. From equation (8.39)

$$X(s) = \frac{x(0)}{(s-a)} + \frac{b}{(s-a)}U(s) \tag{8.40}$$

Inverse transform

$$x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \tag{8.41}$$

where the integral term in equation (8.41) is the convolution integral and τ is a dummy time variable. Note that

$$\mathrm{e}^{at} = 1 + at + \frac{a^2t^2}{2!} + \cdots + \frac{a^kt^k}{k!} \tag{8.42}$$

Consider now the state vector differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.43}$$

Taking Laplace transforms

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{8.44}$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Pre-multiplying by $(s\mathbf{I} - \mathbf{A})^{-1}$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{8.45}$$

Inverse transform

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{U}(\tau)\mathrm{d}\tau \tag{8.46}$$

if the initial time is t_0, then

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(0) + \int_{t_0}^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{8.47}$$

The exponential matrix $\mathrm{e}^{\mathbf{A}t}$ in equation (8.46) is called the state-transition matrix $\mathbf{\Phi}(t)$ and represents the natural response of the system. Hence

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \tag{8.48}$$

$$\mathbf{\Phi}(t) = \mathscr{L}^{-1}(s\mathbf{I} - \mathbf{A})^{-1} = \mathrm{e}^{\mathbf{A}t} \tag{8.49}$$

Alternatively

$$\mathbf{\Phi}(t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^k}{k!} \tag{8.50}$$

Hence equation (8.46) can be written

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{8.51}$$

In equation (8.51) the first term represents the response to a set of initial conditions, whilst the integral term represents the response to a forcing function.

Characteristic equation

Using a state variable representation of a system, the characteristic equation is given by

$$|(s\mathbf{I} - \mathbf{A})| = 0 \tag{8.52}$$

8.2.1 Transient solution from a set of initial conditions

Example 8.6
For the spring–mass–damper system given in Example 8.1, Figure 8.1, the state equations are shown in equation (8.13)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{m} & -\dfrac{C}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{8.53}$$

Given: $m = 1\,\mathrm{kg}$, $\mathrm{C} = 3\,\mathrm{Ns/m}$, $K = 2\,\mathrm{N/m}$, $u(t) = 0$. Evaluate,

(a) the characteristic equation, its roots, ω_n and ζ
(b) the transition matrices $\boldsymbol{\phi}(s)$ and $\boldsymbol{\phi}(t)$
(c) the transient response of the state variables from the set of initial conditions

$$y(0) = 1.0,$$
$$\dot{y}(0) = 0$$

Solution
Since $x_1 = y$ and $x_2 = \dot{y}$, then $x_1(0) = 1.0$, $x_2(0) = 0$.
Inserting values of system parameters into equation (8.53) gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(a)
$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & (s+3) \end{bmatrix} \tag{8.54}$$

From equation (8.52), the characteristic equation is

$$|(s\mathbf{I} - \mathbf{A})| = s(s+3) - (-2) = s^2 + 3s + 2 = 0 \tag{8.55}$$

Roots of characteristic equation

$$s = -1, -2 \tag{8.56}$$

Compare equation (8.55) with the denominator of the standard form in equation (3.43)

$$\begin{aligned} \omega_\mathrm{n}^2 &= 2 \quad \text{i.e} \quad \omega_\mathrm{n} = 1.414\,\mathrm{rad/s} \\ 2\zeta\omega_\mathrm{n} &= 3 \quad \text{i.e} \quad \zeta = 1.061 \end{aligned} \tag{8.57}$$

(b) The inverse of any matrix $\mathbf{A}$ (see equation A2.17) is

$$\mathbf{A}^{-1} = \frac{\text{Adjoint } \mathbf{A}}{\det \mathbf{A}} \tag{8.58}$$

From equation (8.48)

$$\boldsymbol{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

Using the standard matrix operations given in Appendix 2, equation (A2.12)

$$\text{Minors of } \mathbf{\Phi}(s) = \begin{bmatrix} (s+3) & 2 \\ -1 & s \end{bmatrix}$$

$$\text{Co-factors of } \mathbf{\Phi}(s) = \begin{bmatrix} (s+3) & -2 \\ 1 & s \end{bmatrix}$$

The Adjoint matrix is the transpose of the Co-factor matrix

$$\text{Adjoint of } \mathbf{\Phi}(s) = \begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix} \tag{8.59}$$

Hence, from equations (8.58) and (8.48)

$$\mathbf{\Phi}(s) = \begin{bmatrix} \dfrac{(s+3)}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} \tag{8.60}$$

Using partial fraction expansions

$$\mathbf{\Phi}(s) = \begin{bmatrix} \left(\dfrac{2}{s+1} - \dfrac{1}{s+2}\right) & \left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) \\ -2\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) & \left(-\dfrac{1}{s+1} + \dfrac{2}{s+2}\right) \end{bmatrix} \tag{8.61}$$

Inverse transform equation (8.61)

$$\mathbf{\Phi}(t) = \begin{bmatrix} (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}) \end{bmatrix} \tag{8.62}$$

Note that the exponential indices are the roots of the characteristic equation (8.56).

(c) From equation (8.51), the transient response is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{8.63}$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.64}$$

$$\begin{aligned} x_1(t) &= (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ x_2(t) &= -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \end{aligned} \tag{8.65}$$

The time response of the state variables (i.e. position and velocity) together with the state trajectory is given in Figure 8.5.

Example 8.7

For the spring–mass–damper system given in Example 8.6, evaluate the transient response of the state variables to a unit step input using

(a) The convolution integral
(b) Inverse Laplace transforms

Assume zero initial conditions.

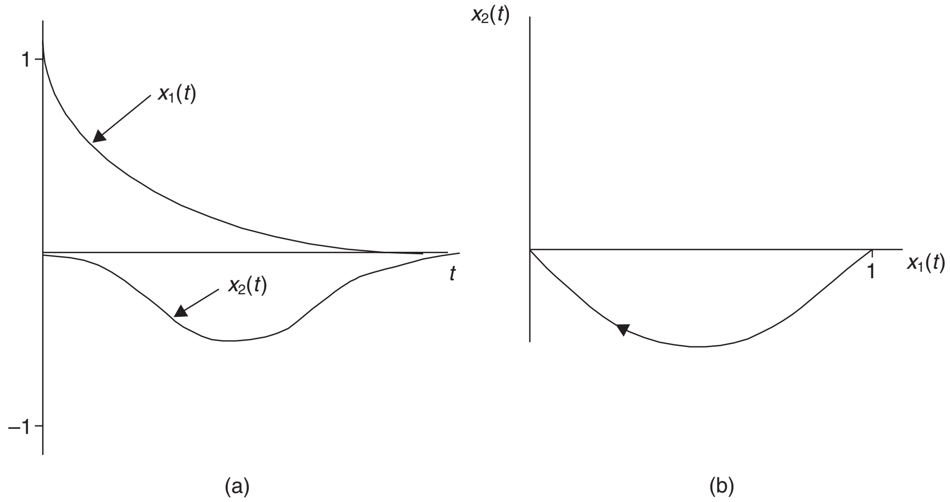


Fig. 8.5 State variable time response and state trajectory for Example 8.4.

Solution

(a) From equation (8.51)

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \mathbf{u}(\tau)\mathrm{d}\tau \tag{8.66}$$

Given that $u(t) = 1$ and $1/m = 1$, equation (8.66) reduces to

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} \phi_{12}(t-\tau) \\ \phi_{22}(t-\tau) \end{bmatrix} \mathrm{d}\tau$$

Inserting values from equation (8.62)

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} \mathrm{e}^{-(t-\tau)} - \mathrm{e}^{-2(t-\tau)} \\ \mathrm{e}^{-(t-\tau)} + 2\mathrm{e}^{-2(t-\tau)} \end{bmatrix} \mathrm{d}\tau \tag{8.67}$$

Integrating

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{-(t-\tau)} - \frac{1}{2}\mathrm{e}^{-2(t-\tau)} \\ \mathrm{e}^{-(t-\tau)} + \mathrm{e}^{-2(t-\tau)} \end{bmatrix}_0^t \tag{8.68}$$

Inserting integration limits ($\tau = t$ and $\tau = 0$)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{-2t} \\ \mathrm{e}^{-t} - \mathrm{e}^{-2t} \end{bmatrix} \tag{8.69}$$

(b) An alternative method is to inverse transform from an s-domain expression. Equation (8.45) may be written

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{U}(s) \tag{8.70}$$

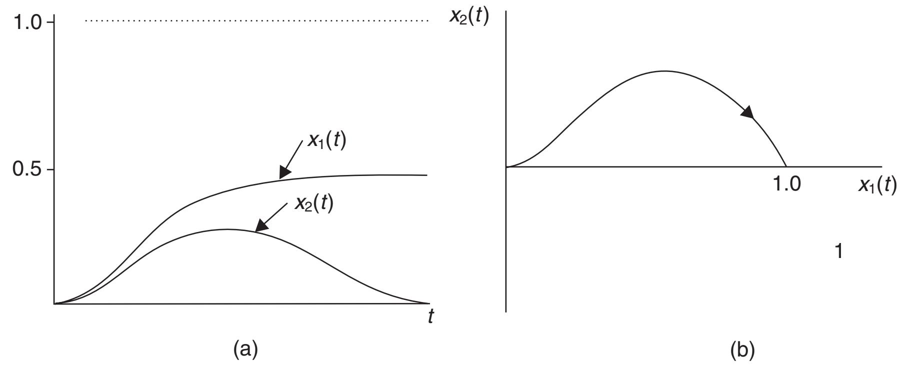


Fig. 8.6 State variable step response and state trajectory for Example 8.5.

Hence from equation (8.61)

$$\mathbf{X}(s) = \boldsymbol{\Phi}(s)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \left(\dfrac{2}{s+1} - \dfrac{1}{s+2}\right) & \left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) \\ -2\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) & \left(\dfrac{-1}{s+1} + \dfrac{2}{s+2}\right) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s} \tag{8.71}$$

Simplifying

$$\mathbf{X}(s) = \begin{bmatrix} \dfrac{1}{s(s+1)} - \dfrac{1}{2}\left\{\dfrac{2}{s(s+2)}\right\} \\ \dfrac{-1}{s(s+1)} + \dfrac{2}{s(s+2)} \end{bmatrix} \tag{8.72}$$

Inverse transform

$$\mathbf{x}(t) = \begin{bmatrix} (1 - \mathrm{e}^{-t}) - \frac{1}{2}(1 - \mathrm{e}^{-2t}) \\ -(1 - \mathrm{e}^{-t}) + (1 - \mathrm{e}^{-2t}) \end{bmatrix} \tag{8.73}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{-2t} \\ \mathrm{e}^{-t} - \mathrm{e}^{-2t} \end{bmatrix} \tag{8.74}$$

Equation (8.74) is the same as equation (8.69).

The step response of the state variables, together with the state trajectory, is shown in Figure 8.6.

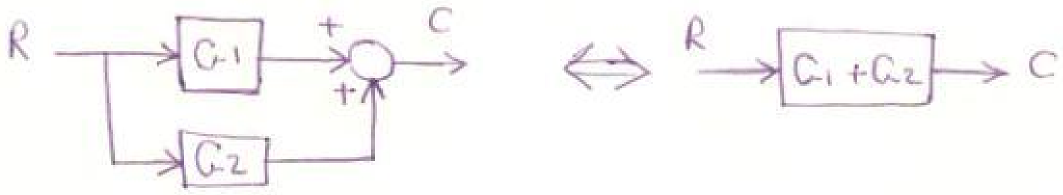
8.3 Discrete-time solution of the state vector differential equation

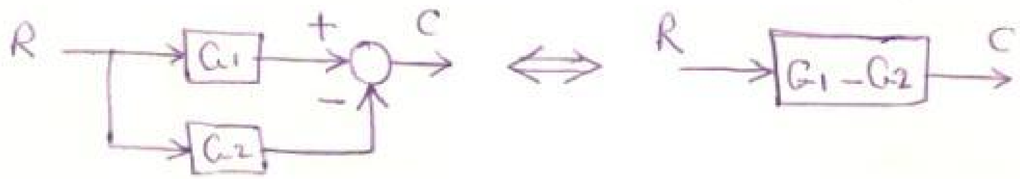
The discrete-time solution of the state equation may be considered to be the vector equivalent of the scalar difference equation method developed from a z-transform approach in Chapter 7.

Block Diagram Reduction

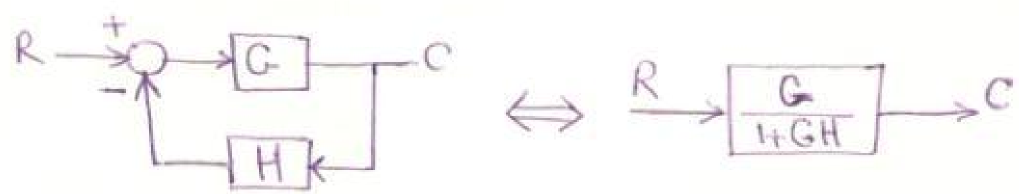
Rule ① Cascaded elements

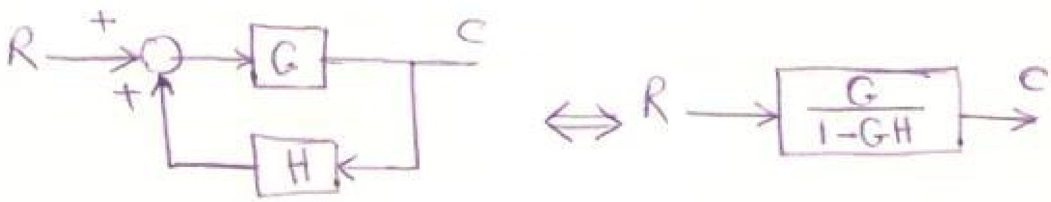
Rule ② Addition or Subtraction



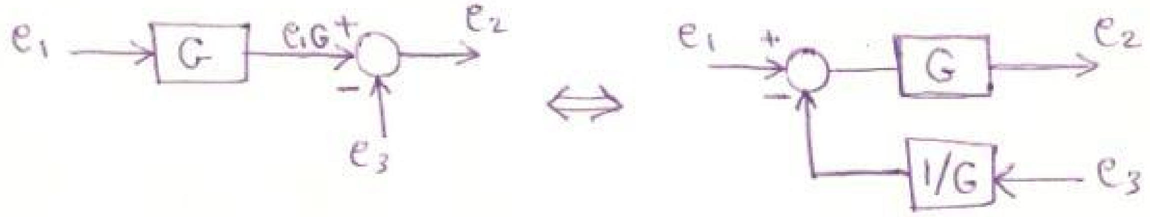


Rule ③ Closed - Loop

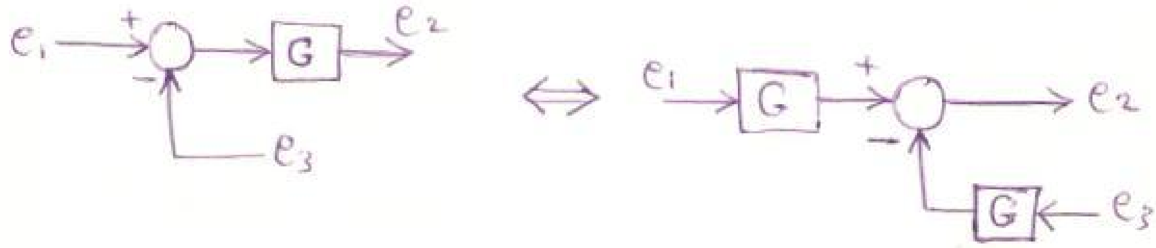




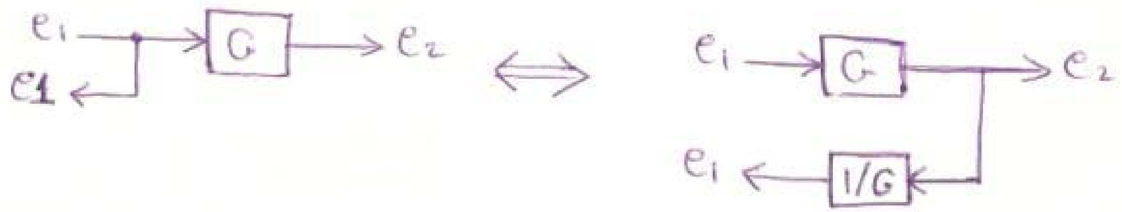
Rule (4) Moving a Summing point a head of a block



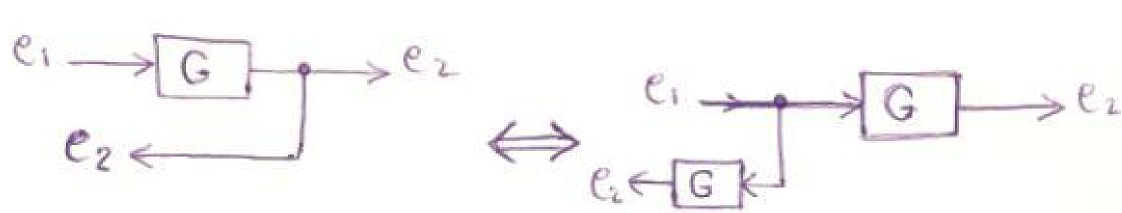
Rule (5) Moving a Summing point behind a block



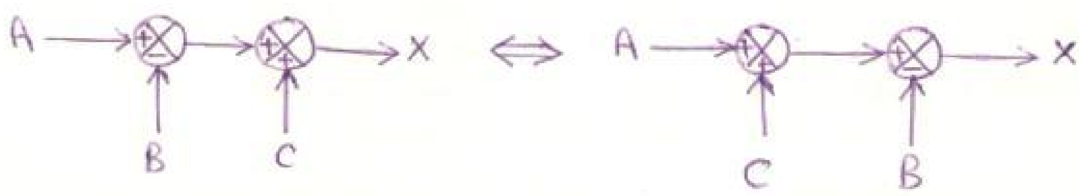
Rule (6) Moving a pick off point behind a block



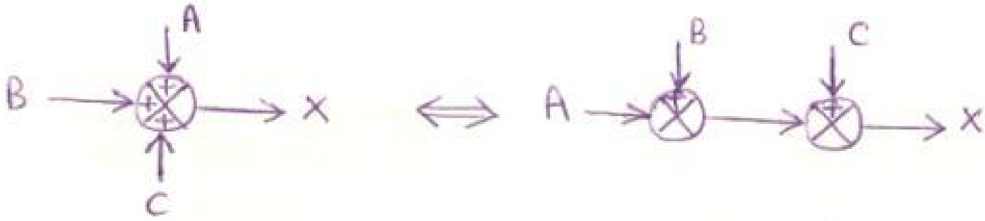
Rule (7) Moving a pick off point a head of a block



Rule 8

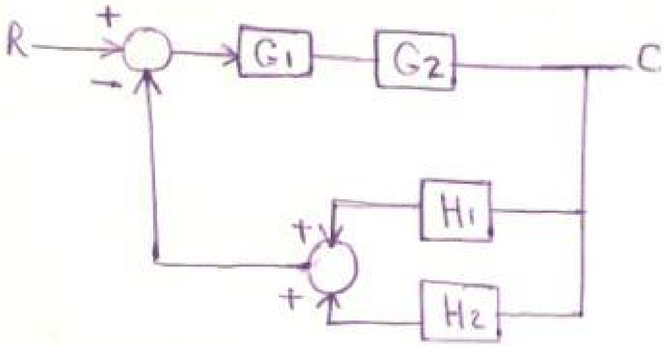


$X = A - B + C$ $\qquad$ $X = A + C - B$

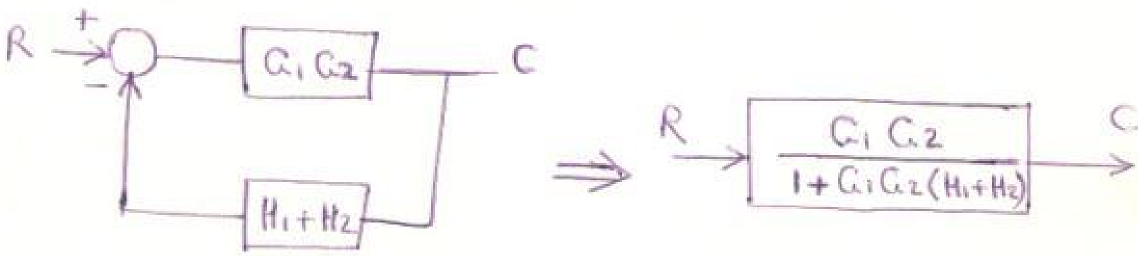


$X = A + B + C$ $\qquad$ $X = A + B + C$

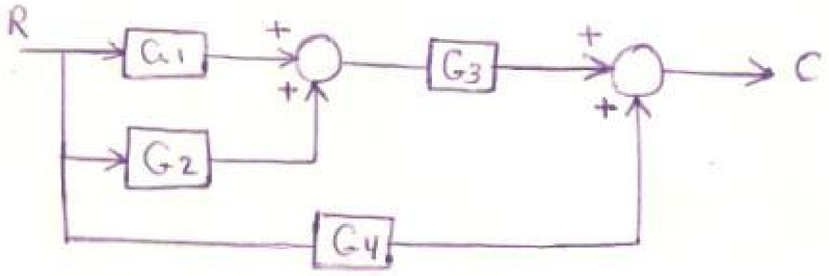
Ex1:



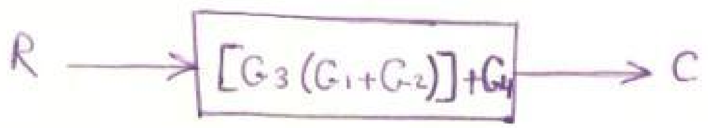
$\Downarrow$



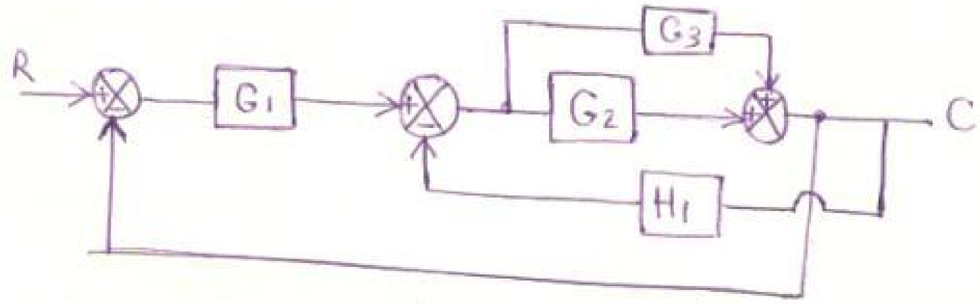
Ex2:


R
G1
G2
G3
G4
C

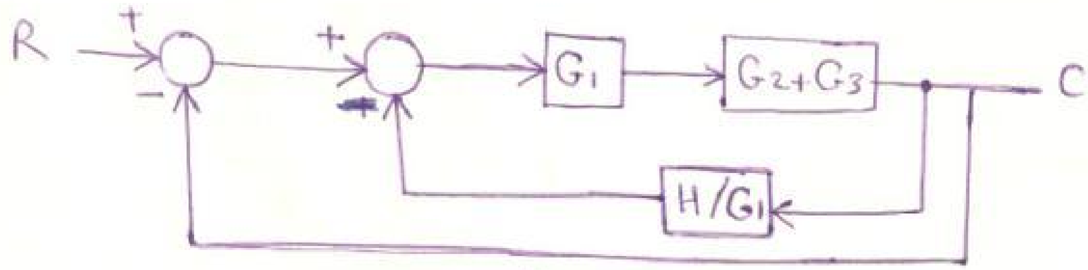
⇓


R
[G3 (G1+G2)]+G4
C

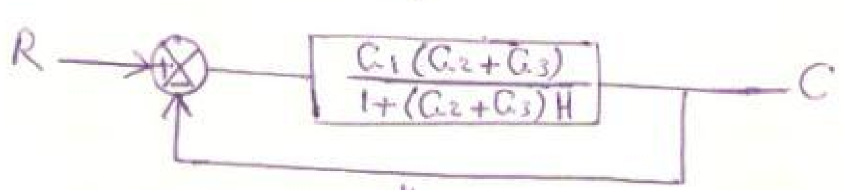
Ex3:-


R
G1
G2
G3
H1
C

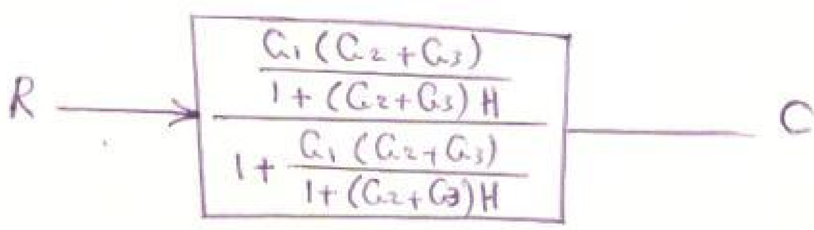
⇓


R
G1
G2+G3
H/G1
C

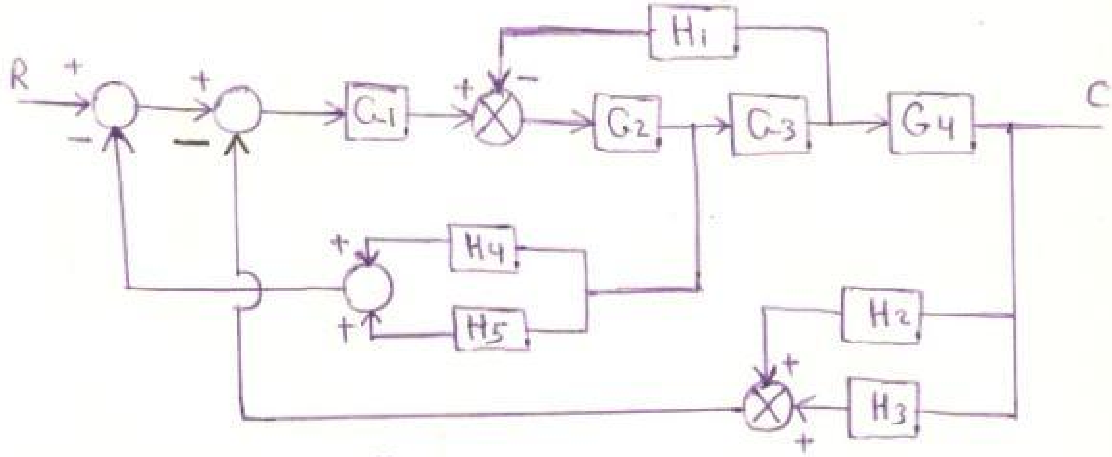
⇓


R
G1 (G2+G3) / 1+ (G2+G3) H
C

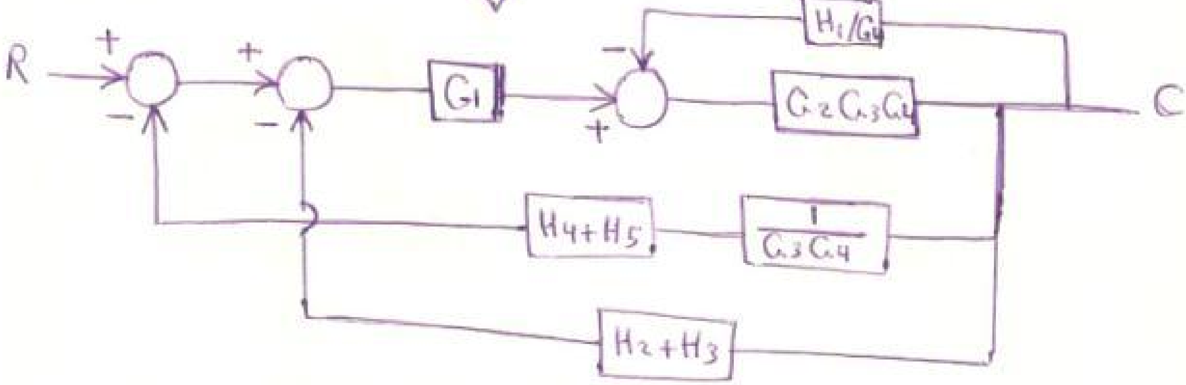
⇓


R
G1 (G2+G3) / 1+ (G2+G3) H
1+ G1 (G2+G3) / 1+ (G2+G3) H
C

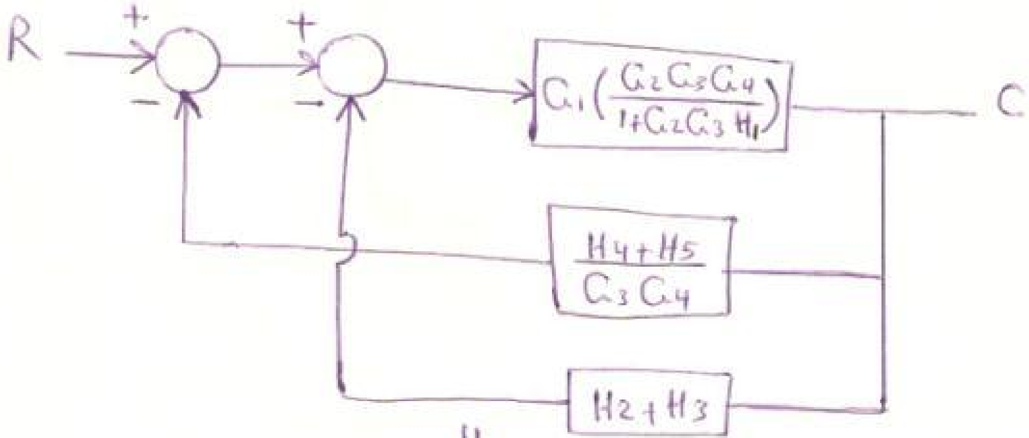
Ex4:



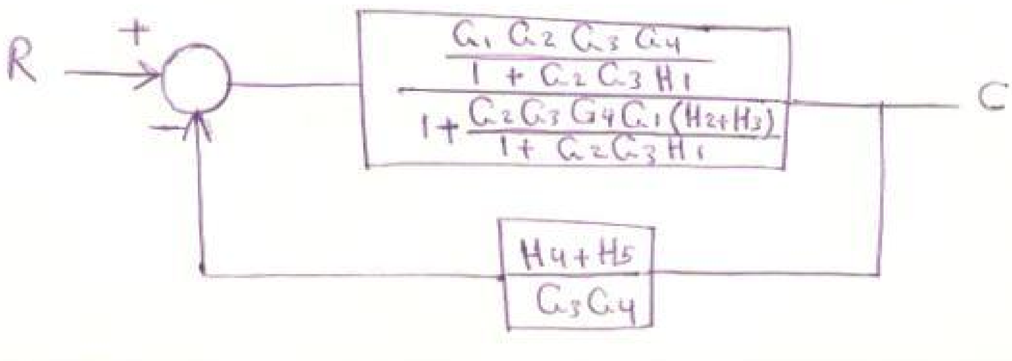
$\Downarrow$

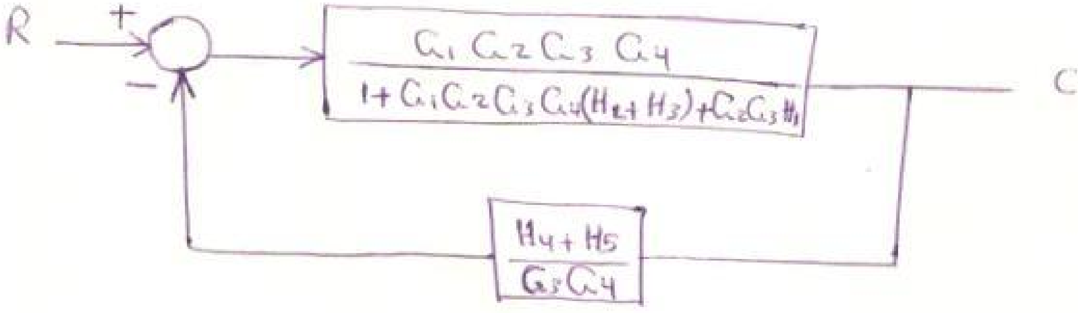


$\Downarrow$

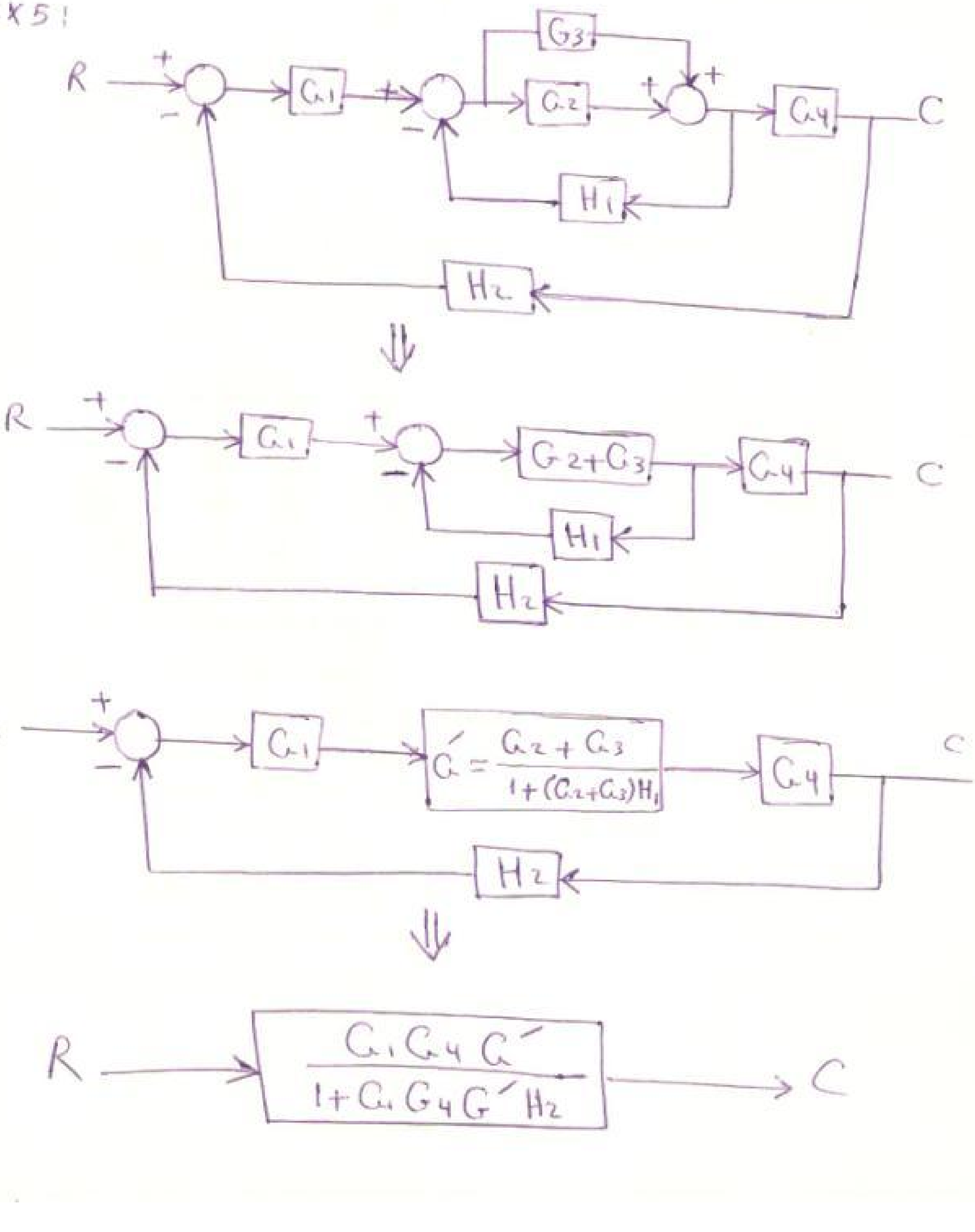


$\Downarrow$



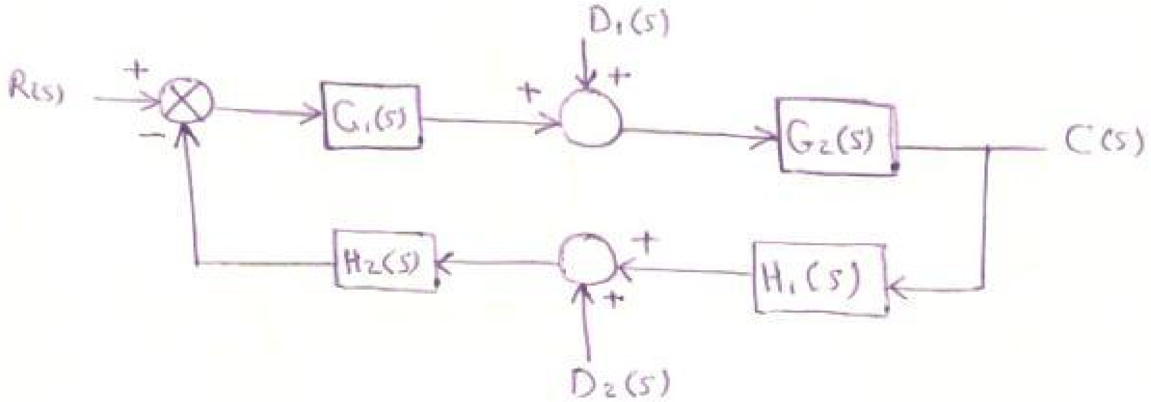

R
+
−
$\frac{G_1 G_2 G_3 G_4}{1+G_1G_2G_3G_4(H_2+H_3)+G_2G_3H_1}$
C
$\frac{H_4+H_5}{G_3G_4}$

Ex5:


R
C
G_1
G_2
G_3
G_4
H_1
H_2
G_2+G_3
$G' = \frac{G_2+G_3}{1+(G_2+G_3)H_1}$
$\frac{G_1 G_4 G'}{1+G_1 G_4 G' H_2}$

The principles of superposition

Ex: Determine the output for the system shown below:

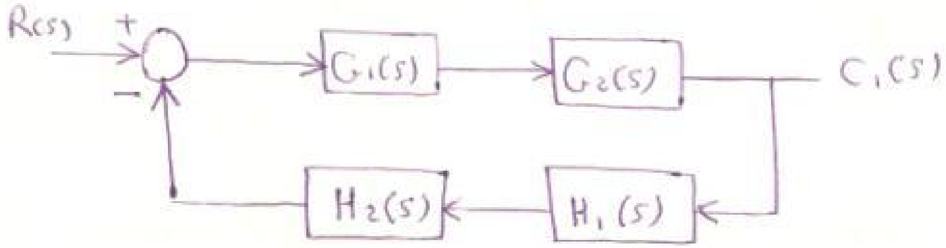


Solution: According to the principle of superposition, then, we must try to find the output considering one input at a time.

① - output due to input $R(s)$

Let $D_1(s) = 0$ & $D_2(s) = 0$

Hence the system becomes:

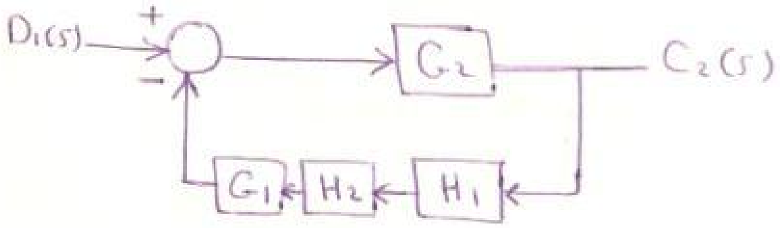


which gives
$$\frac{C_1(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_2(s)\,H_1(s)\,H_2(s)} \quad \text{......} \quad ①$$

(2). output due to input $D_1(s)$

Let $R(s) = 0$ & $D_2(s) = 0$

We expect the system to reduce to

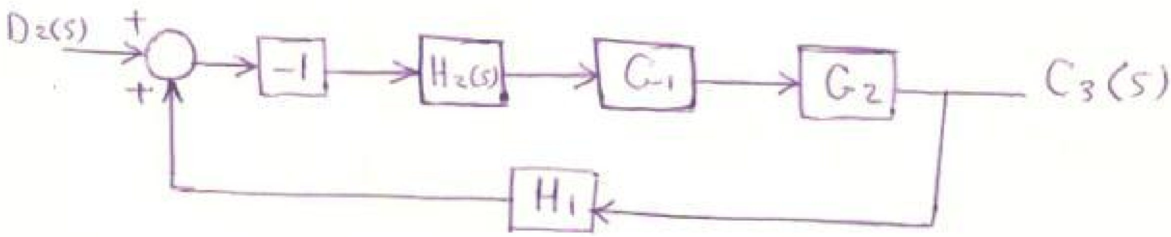


$$\therefore \frac{C_2(s)}{D_1(s)} = \frac{G_2}{1 + G_1 G_2 H_1 H_2} \quad \text{------} \quad (2)$$

(3). output due to input $D_2(s)$

Let $R(s) = 0$ & $D_1(s) = 0$

Hence the system becomes

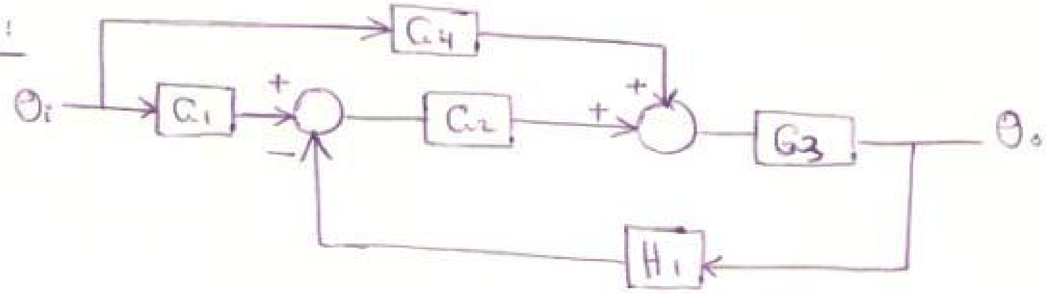


$$\therefore \frac{C_3(s)}{D_2(s)} = \frac{-G_1 G_2 H_2}{1 + G_1 G_2 H_1 H_2} \quad \text{------} \quad (3)$$

$\therefore$ Total output $C = C_1 + C_2 + C_3$

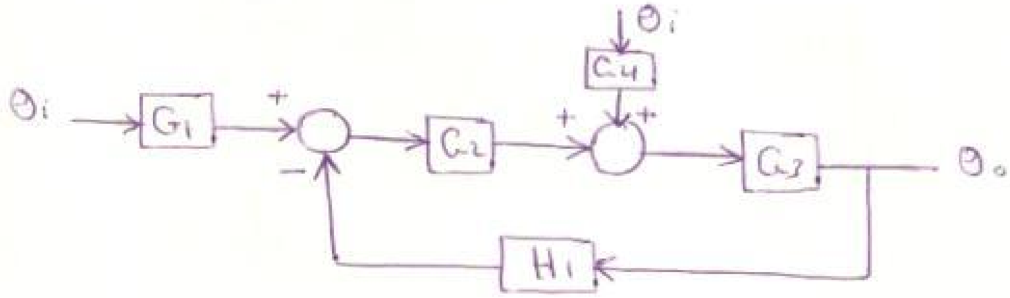
$$\text{i.e } C = \frac{R\, G_1 G_2}{1 + G_1 G_2 H_1 H_2} + \frac{D_1\, G_2}{1 + G_1 G_2 H_1 H_2} - \frac{D_2\, G_1 G_2 H_2}{1 + G_1 G_2 H_1 H_2}$$

Ex 2:



Find the relationship between θ_o & θ_i

Solution: Redraw the diagram as follows:

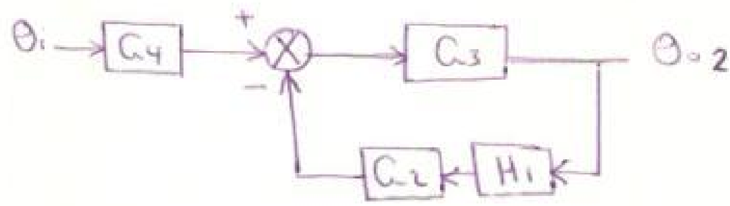


①. Let $\theta_i G_4 = 0$

$$\frac{\theta_{o1}}{G_1\theta_i} = \frac{G_2G_3}{1+G_2G_3H_1}$$

$$\therefore \frac{\theta_{o1}}{\theta_i} = \frac{G_1G_2G_3}{1+G_2G_3H_1} \quad \text{---} ①$$

②. Let $\theta_i G_1 = 0$



$$\frac{\theta_{o2}}{\theta_i G_4} = \frac{G_3}{1+G_3G_2H_1}$$

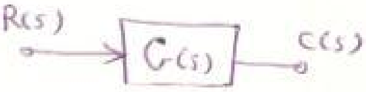
$$\frac{\theta_{o2}}{\theta_i} = \frac{G_3G_4}{1+G_2G_3H_1} \quad \text{---} ②$$

$$\therefore \text{total } \frac{\theta_o}{\theta_i} = \frac{\theta_{o1}}{\theta_i} + \frac{\theta_{o2}}{\theta_i} = \frac{G_1G_2G_3}{1+G_2G_3H_1} + \frac{G_3G_4}{1+G_2G_3H_1} = \frac{G_3(G_1G_2+G_4)}{1+G_2G_3H_1}$$

Signal Flow Graph

It is a method for presenting a control system, it is Like a network, it consists of: Junction points called (Nodes) and Directed Line segments Called (Branches).

Suppose we have



$C(s) = G(s)\,R(s)$

$C = GR$

The direction of the branch is from R to C.

where

C .. is the output Node.

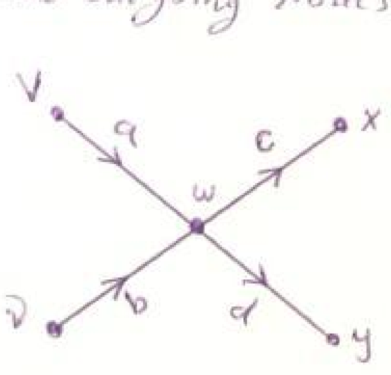
R .. is the input Node.

G .. is called as the transmittance or Gain.

There are different types of Nodes:

Source Nodes .... outgoing Nodes.
Sink Nodes .... incomming Nodes.
mixed Nodes .... incomming and outgoing Nodes.

V, ϑ .. are Source nodes.
W .. mixed nodes.
X, y .. are sink nodes.

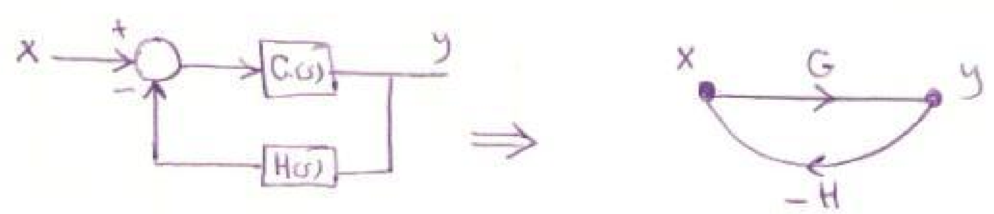


$w = av + bv$

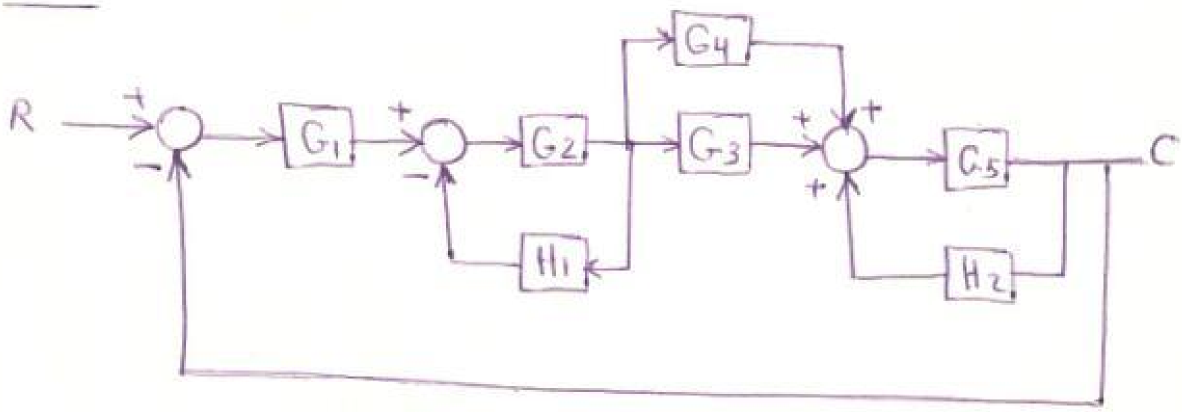
$x = cw$

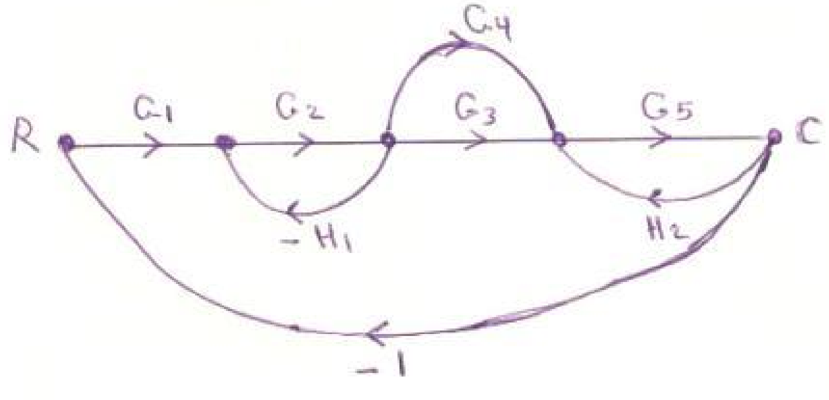
$y = dw$

Also, the closed Loop control system represented as follows:



Ex:





Paths

1 - $G_1G_2G_3G_5$

2 - $G_1G_2G_4G_5$

Loops

1 - $G_1G_2G_3G_5(-1)$

2 - $G_1G_2G_4G_5(-1)$

3 - $-G_2H_1$

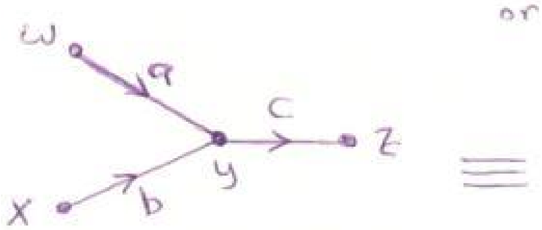
4 - G_5H_2

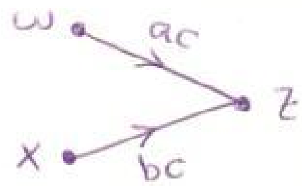
Some of the rules governing flow Graphs

(a) Series paths (These may be Combined)

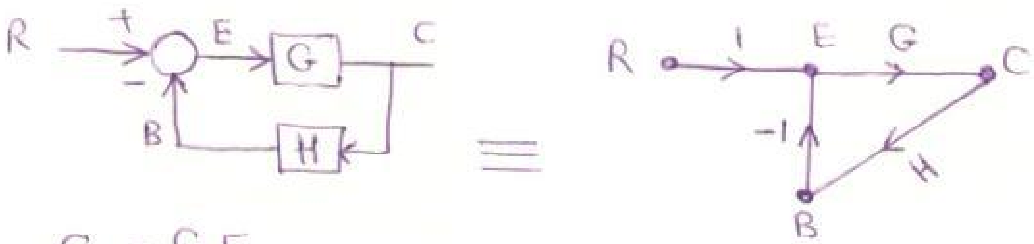
(b) Parallel paths (Addition of transmittance or Gain)

(c) Node absorption : It is a node which represented a Variable which is neither a source or a Sink. This may be elliminated.





(d) Feedback paths

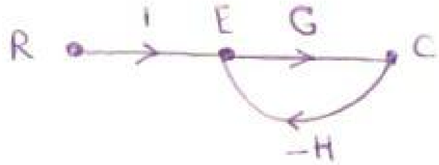


$C = GE$

$B = HC$

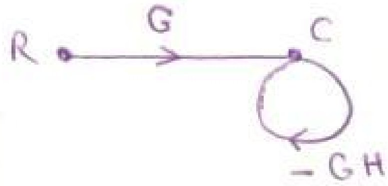
$E = R - B$

Elliminate B we get

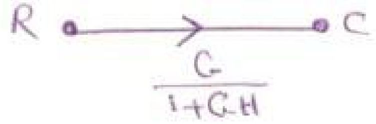


Elliminate E we get

$C = G(R - B)$
$= GR - GHC$



The Final Simplification is



The over all transmittance or T.F canbe obtained using Mason's Rule.

Mason's Rule

It is an algorithim use to Calculate the total transfer function T(s) for a control system represented by a signal Flow graph according to the equation:

$$T(s) = \frac{1}{\Delta} \sum_{i=1}^{n} P_i \, \Delta i$$

Where

T... is the total T.F of a system.

Pi -- Path gain or transmittance of ith forward path.

Δ -- determinant of graph.

$$\Delta = 1 - \sum_{i=1}^{n} L_i + \sum_{i,j=1}^{n} L_i L_j - \sum_{i,j,k=1}^{n} L_i L_j L_k + \sum \cdots$$

Where

$\sum_{i=1}^{n} L_i$ .... Sum of all individual Loop gains.

$\sum_{i,j=1}^{n} L_i L_j$ ... Sum of gain products of all possible Combinations of two non touching Loops.

$\sum_{i,j,k=1}^{n} L_i L_j L_K$ .... Sum of gain products of all possible Combinations of three nontouching Loops.

Δi ... Cofactor of the ith forward path determinant of the graph with the Loops touching the ith forward path removed, that is the cofactor Δi is obtained from Δ by removing the Loops that touch path P_i.

Note

* The summations are kaken over all possible paths from input to output.
* In Δ the multiplication of touch loop equal Zero.

Ex: Find Δ we have L_1, L_2, L_3 that not touch each other.

$$\sum_{i=1}^{3} L_i = L_1 + L_2 + L_3$$

$$\sum_{\substack{i=1 \\ j=1}}^{3} L_i L_j = L_1 L_2 + L_1 L_3 + L_2 L_3$$

$$\sum_{\substack{i=1 \\ j=1 \\ k=1}}^{3} L_i L_j L_K = L_1 L_2 L_3$$

$$\therefore \Delta = 1 - (L_1 + L_2 + L_3) + L_1L_2 + L_1L_3 + L_2L_3 - (L_1L_2L_3)$$

Ex: Find Δ we have L_1, L_2, L_3, L_4 that non touch.

$$\sum_{i=1}^{4} L_i = L_1 + L_2 + L_3 + L_4$$

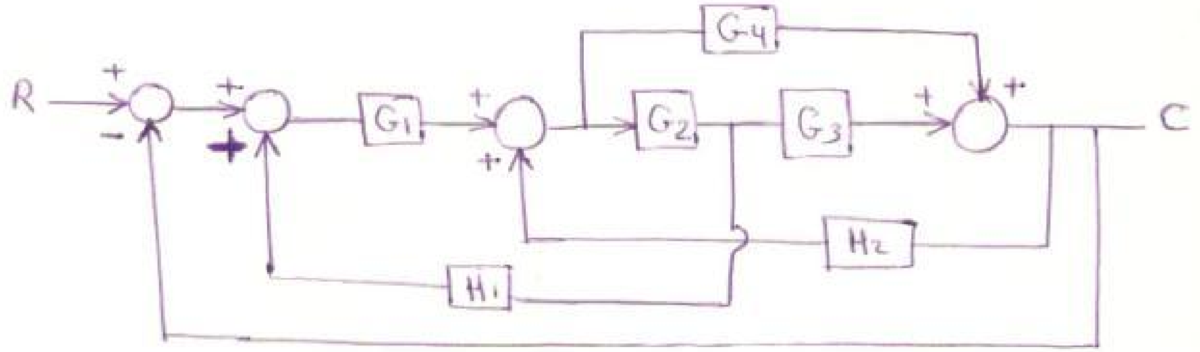
$$\sum_{\substack{i=1\\j=1}}^{4} L_iL_j = L_1L_2 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4 + L_3L_4$$

$$\sum_{i,j,k=1}^{4} L_iL_jL_k = L_1L_2L_3 + L_1L_2L_4 + L_2L_3L_4 + L_1L_3L_4$$

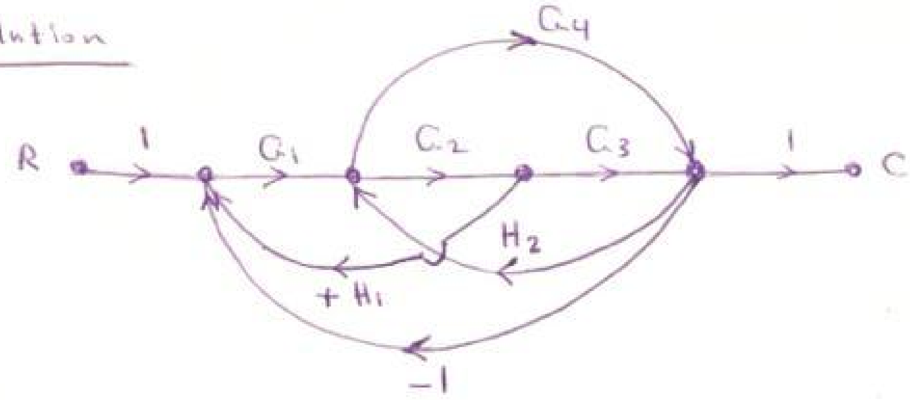
$$\sum_{i,j,k,m=1}^{4} L_iL_jL_kL_m = L_1L_2L_3L_4$$

$$\therefore \Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1L_2 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4 + L_3L_4 - (L_1L_2L_3 + L_1L_2L_4 + L_2L_3L_4 + L_1L_3L_4) + L_1L_2L_3L_4$$

Ex_1: Find the T.F $\left(\frac{C}{R}\right)$ for the control system given by:



Solution



Paths

$P_1 = G_1 G_2 G_3$

$P_2 = G_1 G_4$

Loops

$L_1 = +\, G_1 G_2 H_1$

$L_2 = -\, G_1 G_2 G_3$

$L_3 = G_2 G_3 H_2$

$L_4 = -\, G_1 G_4$

$L_5 = G_4 H_2$

$$\sum_{i=1}^{5} L_i = L_1 + L_2 + L_3 + L_4 + L_5$$
$$= G_1G_2H_1 - G_1G_2G_3 + G_2G_3H_2 - G_1G_4 + G_4H_2$$

$$\sum_{\substack{i=1\\j=1}}^{5} L_iL_j = L_1L_2 + L_1L_3 + L_1L_4 + L_1L_5 + L_2L_3 + L_2L_4 + L_2L_5 + L_3L_4 + L_3L_5 + L_4L_5$$
$$= 0$$

$$\sum_{i,j,k=1}^{5} L_iL_jL_k = 0$$

$$\sum_{i,j,k,m=1}^{5} L_iL_jL_kL_m = 0$$

$$\sum_{i,j,k,m,n=1}^{5} L_iL_jL_kL_mL_n = 0$$

①- Forward path P_1

$\Delta_i = 1 - \Sigma + \Sigma - \Sigma$

$\Delta_1 = 1$

$P_1\Delta_1 = G_1G_2G_3$

②- Forward path P_2

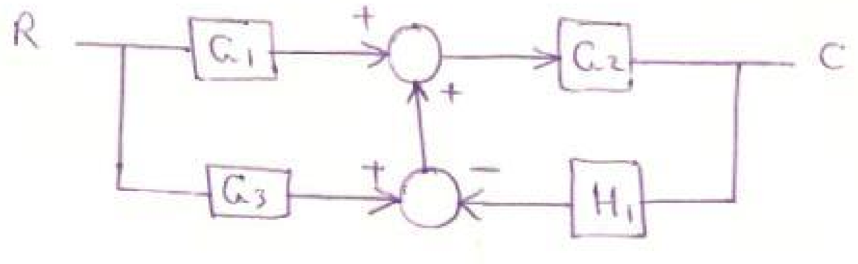
$\Delta_2 = 1$

$P_2\Delta_2 = G_1G_4$

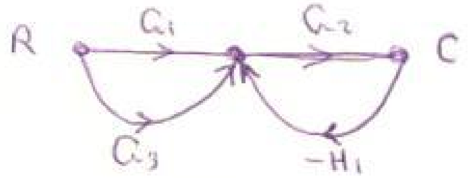
$$\therefore \frac{C}{R} = \frac{\sum P_i\Delta_i}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1 - G_1G_2H_1 + G_1G_2G_3 - G_2G_3H_2 + G_1G_4 - G_4H_2}$$

Ex₂: Determine $\left(\frac{C}{R}\right)$ for the system given by:



Solution



Paths:

$P_1 = G_1 G_2$

$P_2 = G_3 G_2$

Loops:

$L_1 = - G_2 H_1$

$\Delta = 1 - (- G_2 H_1)$

$\Delta_1 = 1$

$\Delta_2 = 1$

$$\frac{C}{R} = \frac{\sum P_i \Delta_i}{\Delta} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_2 + G_3 G_2}{1 + G_2 H_1}$$

Time - Domain Analysis of control system

The time response of a control system consists of two parts : the transient response and the steady - state response. By transient response, we mean that which goes from the initial state to the final state. By steady - state response, we mean the manner in which the system output behaves as (t) approaches infinity. Thus the system response c(t) may be written as

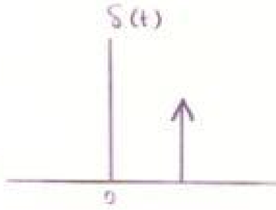
$$c(t) = C_{tr}(t) + C_{ss}(t)$$

where

$C_{tr}(t)$ .... transient response.
$C_{ss}(t)$ .... steady - state response.

The typical test signals for the time response of control system are :

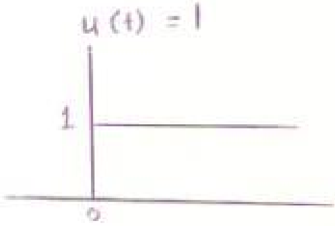
① - Unit Impulse Response : $\delta(s) = 1$



the time response is the inverse Laplace transform of G(s)

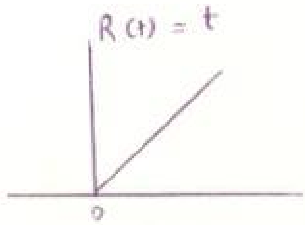
$$y(t) = \mathcal{L}^{-1} G(s)$$

② . Unit step input : $U(s) = \frac{1}{s}$



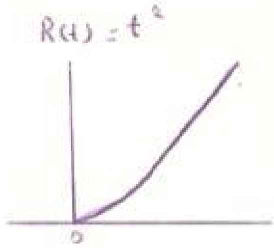
$$y(t) = \mathcal{L}^{-1} G(s) \cdot U(s)$$

③ - Ramp input : $R(s) = \frac{1}{s^2}$



$$y(t) = \mathcal{L}^{-1} G(s) \cdot R(s)$$

④ - Parabolic input : $R(s) = \frac{1}{s^3}$

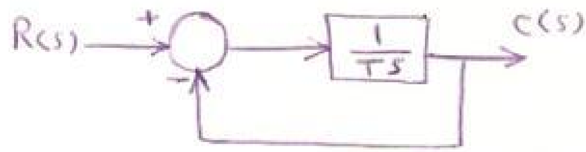


$$y(t) = \mathcal{L}^{-1}\, G(s) \cdot R(s)$$

Transient Response

① - Transint Response of first order system :-

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS}$$



First order system

(a) Unit-step Response : $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{TS+1} \cdot \frac{1}{S}$$

By using partial fraction gives

$$C(s) = \frac{1}{S} - \frac{T}{TS+1} = \frac{1}{S} - \frac{1}{S+\left(\frac{1}{T}\right)}$$

Taking the inverse Laplace transform gives

$$c(t) = 1 - e^{-(t/T)} \quad \text{for } t \geq 0$$

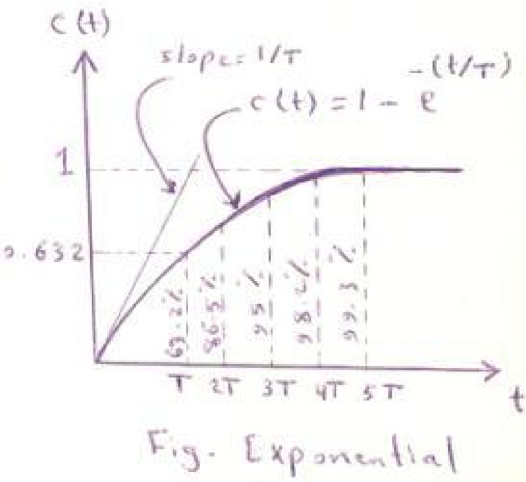


Fig. Exponential response curve

Notes :

1. c(t) initially zero and finally one.
2. At $t = T$, the value of $c(t)$ is 0.632 or c(t) reach 63.2% of its final value, where T is time constant of the system.
3. The exponential response curve is that the slope of the tangent line at $t=0$ is $1/T$.

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T} e^{-t/T}\right|_{t=0} = 1/T$$

(b) Unit-Ramp Response : $R(s) = \frac{1}{s^2}$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2}$$

by partial fraction gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

taking inverse Laplace transform, we obtain

$$c(t) = t - T + T e^{-t/T}, \quad \text{for } t \geq 0$$

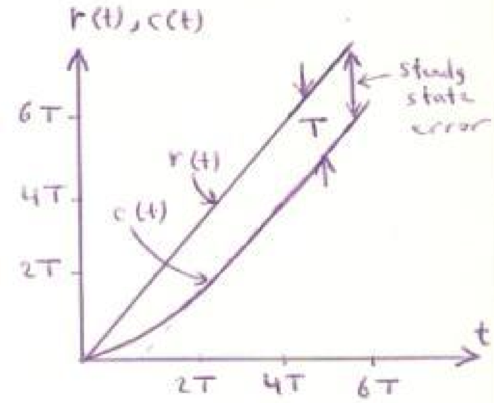


Fig. Unit-ramp response

The error signal $e(t)$ is then

$$e(t) = r(t) - c(t) = T(1 - e^{-t/T})$$

$$e(\infty) = T$$

(c) Unit-Impulse Response : $R(s) = 1$

$$C(s) = \frac{1}{Ts+1}$$

The inverse Laplace transform gives

$$c(t) = \frac{1}{T} e^{-t/T}, \quad \text{for } t \geq 0$$

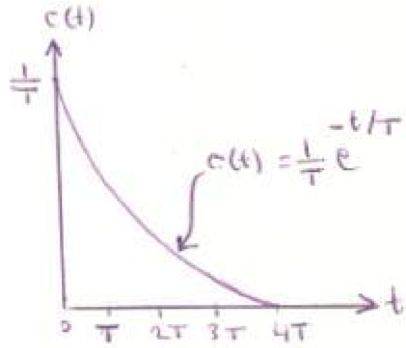


Fig. Unit-impulse response

② - Transient Response of second order system:-

For servo system

$$J\ddot{c} + B\dot{c} = T$$

$$J s^2 c(s) + B s c(s) = T(s)$$

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js+B)}$$

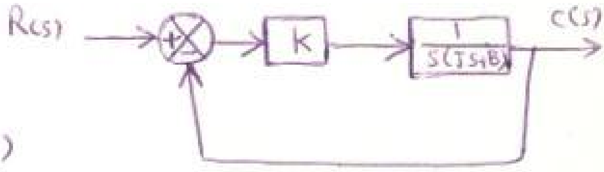


Fig. Servo system

The closed-Loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+Bs+K} = \frac{(K/J)}{s^2+(B/J)s+(K/J)}$$

This T.F can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{K/J}{\left[s+\frac{B}{2J}+\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]\left[s+\frac{B}{2J}-\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2-4JK<0$ and they are real if $B^2-4JK \geq 0$.

In transient response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2 , \quad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where σ -- is called the attenuation

ω_n -- the undamped natural frequency

ζ -- the damping ratio.

The damping ratio (ζ) is the ratio of the actual damping (B) to the critical damping $B_c = 2\sqrt{JK}$

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

hence the T.F can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

This form is called the standard form of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n. If $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half s-plane. The system is then called underdamped, and the transient response is oscillatory. If $\zeta = 0$, the transient response does not die out. If $\zeta = 1$, the system is called critically damped. Overdamped systems corresponed to $\zeta > 1$.

We shall now solve for the response of the second order system (servo system) to a unit-step input for three cases: the underdamped ($0 < \zeta < 1$), critically ($\zeta = 1$), and overdamped ($\zeta > 1$).

(a) Underdamped case ($0 < \zeta < 1$):

In this case, $\frac{C(s)}{R(s)}$ can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where

$\omega_d = \omega_n\sqrt{1-\zeta^2}$. The frequency ω_d is called damped natural frequency.

For a unit-step input, $C(t)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

In inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

hence, $\mathcal{L}^{-1}[C(s)] = c(t)$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \text{ for } t \geq 0$$

$$e(t) = r(t) - c(t)$$

$$= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \text{ for } t \geq 0$$

If $\zeta = 0$,

$$c(t) = 1 - \cos\omega_n t, \text{ for } t \geq 0$$

V

(b) Critically damped case ($\zeta = 1$):

If the two poles of $\frac{C(s)}{R(s)}$ are equal, the system is said to be a critically damped one.

For a unit-step input, $R(s) = \frac{1}{s}$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s}$$

Taking inverse Laplace transform gives

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t), \quad \text{for } t \geq 0$$

(c) Overdamped case ($\zeta > 1$):

In this case, the two poles of $\frac{C(s)}{R(s)}$ are negative real and unequal.

For a unit-step input, $R(s) = \frac{1}{s}$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})\,s}$$

The inverse Laplace transform for above T.F is

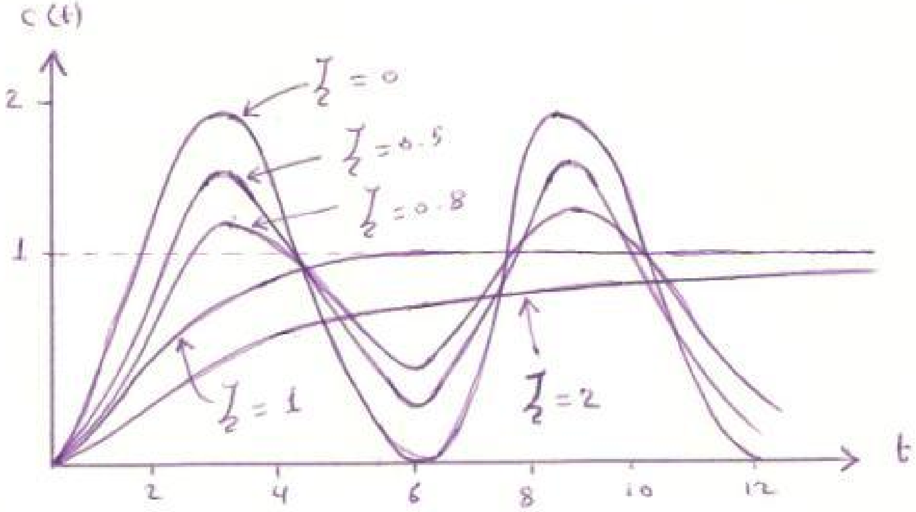
$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})}\, e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}$$

$$- \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})}\, e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \quad \text{for } t \geq 0$$

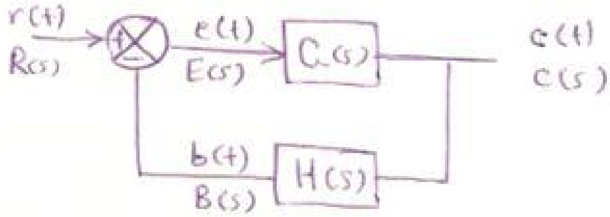
where $s_1 = (\zeta+\sqrt{\zeta^2-1})\omega_n$ and $s_2 = (\zeta-\sqrt{\zeta^2-1})\omega_n$

Thus, the response $c(t)$ includes two decaying exponential terms.



Steady - state Error

The steady-state error is a measure of accuracy and in designed problem one of the objected is to keep this error to minimum value.



$$e(t) = r(t) - b(t)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)\,C(s)$$
$$= R(s) - H(s)\,G(s)\,E(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

steady-state error $= e_{ss} = \lim_{t \to \infty} e(t)$

using final value theorem, e_{ss} is

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} S \cdot E(s) = \lim_{s \to 0} \frac{S\,R(s)}{1 + G(s)\,H(s)}$$

Types of Control system :

In general $G(s)\,H(s)$ may be written as

$$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots\cdots(1+T_m s)}{s^J(1+T_a s)(1+T_b s)\cdots\cdots(1+T_n s)}$$

where K, Ts -- are constants

J ---- type of the system

$J = 0 \Rightarrow$ type 0
$J = 1 \Rightarrow$ type 1
$J = 2 \Rightarrow$ type 2

Ex:

$$G(s)\,H(s) = \frac{K(s+1)}{(s+2)(s+6)} \Rightarrow \text{type } 0$$

$$G(s)\,H(s) = \frac{K}{(s^2+2s)} \Rightarrow \text{type } 1$$

static Position Error Constant K_p ;

The steady-state error of the system for a unit-step input is :

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)\,H(s)} \cdot \frac{1}{s} = \frac{1}{1+\lim_{s\to 0} G(s)\,H(s)}$$

The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s)\,H(s)$$

$$\therefore\ e_{ss} = \frac{1}{1+K_p}$$

for type 0 system

$$K_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots} = K$$

For a type 1 or higher system

$$K_p = \lim_{s \to 0} \frac{K(1+T_1 s)(1+T_2 s) \cdots}{s^J (1+T_a s)(1+T_b s) \cdots} = \infty \quad , \text{for } J \geq 1$$

$$\therefore \quad e_{ss} = \frac{1}{1+K} \quad \text{for typ 0 systems}$$

$$e_{ss} = 0 \quad \text{for type 1 or higher systems}$$

Static velocity Error Constant K_v :

The steady-state error of the system with a unit ramp input is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s) H(s)} \cdot \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{s + s\, G(s) H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s\, G(s)\, H(s)}$$

The static Velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} s\, G(s)\, H(s)$$

$$\therefore \quad e_{ss} = \frac{1}{K_v}$$

For a type 0 system, $K_v = \lim_{s \to 0} \frac{sK(1+T_1 s)(1+T_2 s) \cdots}{(1+T_a s)(1+T_b s) \cdots} = 0$

For a type 1 system,

$$K_v = \lim_{s \to 0} \frac{sK(1+T_1 s)(1+T_2 s) \cdots}{s(1+T_a s)(1+T_b s) \cdots} = K$$

For a type 2 or higher system,

$$K_v = \lim_{s \to 0} \frac{sK(1+T_1 s)(1+T_2 s) \cdots}{s^J (1+T_a s)(1+T_b s) \cdots} = \infty \quad , \text{for } J \geq 2$$

$$\therefore \; e_{ss} = \frac{1}{K_v} = \infty \quad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \quad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0 \quad , \text{for type 2 or higher systems}$$

Static Acceleration Error Constant K_a :

The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)\,H(s)} \cdot \frac{1}{s^3} = \frac{1}{\lim_{s \to 0} s^2 G(s)\,H(s)}$$

$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s)$$

$$e_{ss} = \frac{1}{K_a}$$

For a type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots} = 0$$

For a type 1 system

$$K_a = \lim_{s \to 0} \frac{s^2 k\,(1+T_1 s)(1+T_2 s)\cdots}{s(1+T_a s)(1+T_b s)\cdots} = 0$$

For a type 2 system

$$K_a = \lim_{s \to 0} \frac{s^2 k\,(1+T_1 s)(1+T_2 s)\cdots}{s^2(1+T_a s)(1+T_b s)\cdots} = K$$

For a type 3 or higher system

$$K_a = \lim_{s \to 0} \frac{s^2 k\,(1+T_1 s)(1+T_2 s)\cdots}{s^J(1+T_a s)(1+T_b s)\cdots} = \infty, \quad \text{for } J \geqslant 3$$

$\therefore$ $e_{ss} = \infty$, for type 0 and type 1 systems

$e_{ss} = \frac{1}{K}$, for type 2 systems

$e_{ss} = 0$, for type 3 or higher systems

type	K_p	K_v	K_a	e_{ss} step	e_{ss} Ramp	e_{ss} Parabolic
0	K	0	0	$\frac{1}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{1}{K}$	∞
2	∞	∞	K	0	0	$\frac{1}{K}$
3	∞	∞	∞	0	0	0

Table – K_p, K_v, K_a, and steady-state error in terms of Gain K.

Definitions of Transient - Response Specifications

① - Delay time t_d : The delay time is the time required for the response to reach half the final value the very first time.

② - Rise time t_r : The rise time is the time required for the response to rise from 10% to 90% of its final value.

③ - Peak time t_p : The peak time is the time required for the response to reach the first peak of the overshot.

④ - Maximum (percent) overshot M_p : The maximum overshot is the maximum peak value of the response curve measured from unity.

$$\text{Maximum percent overshot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100\%$$

⑤ - Settling time t_s : The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).

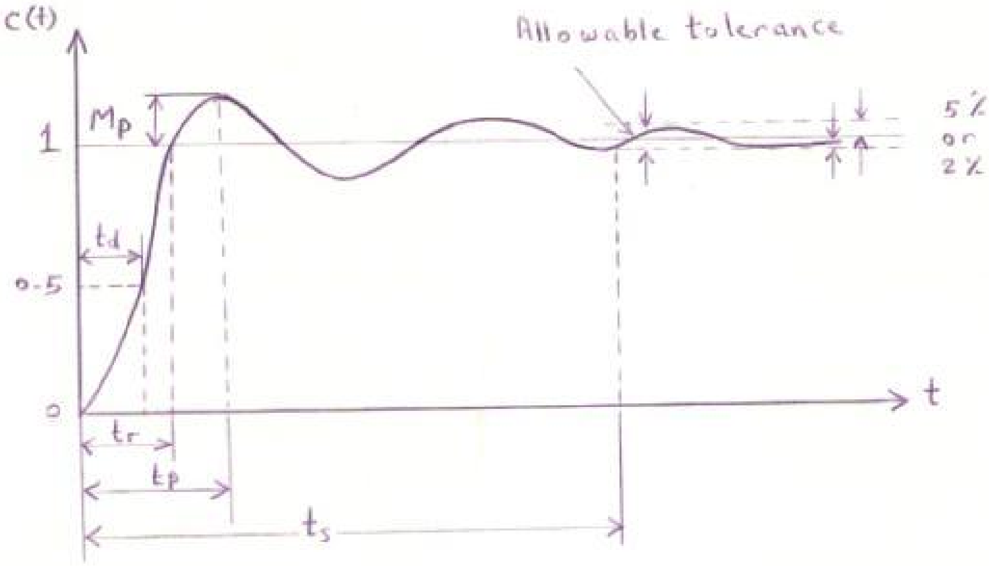


Fig. Unit-step response curve showing t_d, t_r, t_p, M_p, and t_s.

In the following, we shall obtain the rise time, peak time, maximum overshot, and settling time of the second-order system given by equation:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The system is assumed to be underdamped.

<u>Rise time t_r:</u> Referring to equation:

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

We obtain the rise time t_r by letting $c(t_r) = 1$

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

Since $e^{-\zeta\omega_n t_r} \neq 0$, we obtain

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

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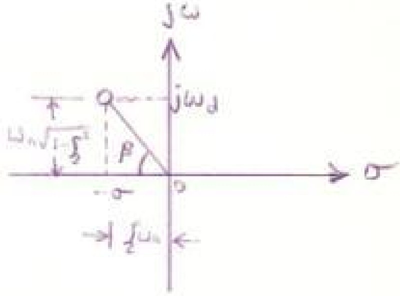
or $\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$

$$\therefore \ t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\sigma}\right)$$

$$t_r = \frac{\pi - \beta}{\omega_d}$$



Peak time t_p : Referring to equation :

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t\right)$$

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t\right) +$$

$$+ e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} \cos\omega_d t\right)$$

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = (\sin\omega_d t_p) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

$$\sin\omega_d t_p = 0$$

or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$

Since the peak time corresponds to the first peak overshot:

$$\omega_d t_p = \pi$$

hence $t_p = \frac{\pi}{\omega_d}$

17

Maximum overshot M_p :

The maximum overshot occurs at the peak time or at $t = t_p = \frac{\pi}{\omega_d}$.
Assuming that the final value of the output is unity.

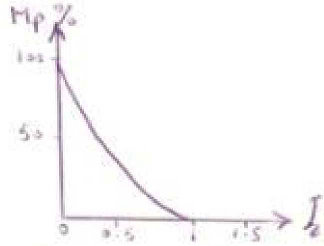
$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} = e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi}$$

The maximum percent overshot is $e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} \times 100\,\%$

If the final value $c(\infty)$ of the output is not unity, then

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

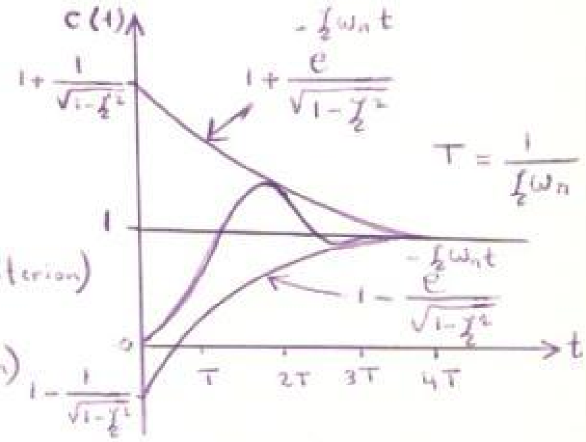


Settling time t_s :

For an undamped second-order system, the transient response is obtained from equation:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \quad \text{for } t \geq 0$$

The curves $1 \mp \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)$ are the envelope curves of the transient response to a unit-step input.



$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (2\% \text{ criterion})$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (5\% \text{ criterion})$$

IV

Ex: Consider the system shown in figure, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec.

Find t_r, t_p, M_p, and t_s when the system is subjected to a unit-step input.

Solution:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-(0.6)^2} = 4$$

$$\sigma = \omega_n \zeta = (5)(0.6) = 3$$

① $$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14-\beta}{4}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

$$\therefore t_r = \frac{3.14-0.93}{4} = 0.55 \text{ sec}$$

② $$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

③ $$M_p = e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} = e^{-\left(\frac{3}{4}\right)* 3.14} = 0.095$$

The maximum percent over shot is thus 9.5 %

④ For the 2% criterion

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5 % criterion

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Stability of Control System

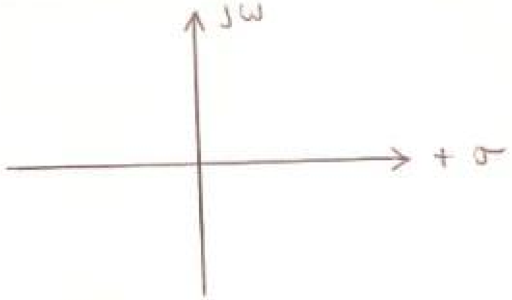
Complex plane

The complex quantity

$$S = \sigma + j\omega$$

Where σ and ω are real variables.

The plane in which the real axis is represented by (σ) and the imajenary axis is represented by (ω) is refered to as the complex plane or the s-plane.



Poles and Zeros

Most T.F.s are expressed in terms of (S), as a ratio of two polynomials, i.e

$$T.F = \frac{(S-z_1)(S-z_2)\,---}{(S-P_1)(S-P_2)\,---} = F(s)$$

Each value of (S) which makes F(s) zero is called as a zero of F(s).

And each value of (S) which makes F(s) infinity is called a pole of F(s).

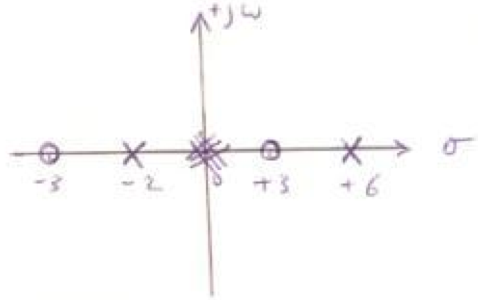
Poles are usually represented by (x).

Zeros are usually represented by (o).

Ex. Find the poles and zeros of F(s).

where $$F(s) = \frac{s^2 - 9}{s^3(s+2)(s-6)} = \frac{(s-3)(s+3)}{s^3(s+2)(s-6)}$$

There are two zeros and five poles (three of which are at the origin).

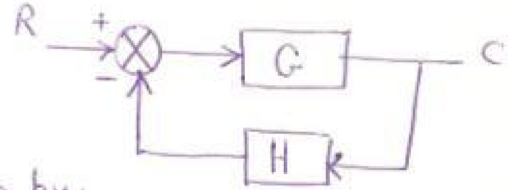


Characteristic Equation

It is the equation formed by puting the denominator of the T.F. of the system equal to zero.

i.e

$$C/L.\ T.F = \frac{G}{1+GH}$$



the C.E of this system is given by:

$$1 + GH = 0$$

If $$T.F = \frac{a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

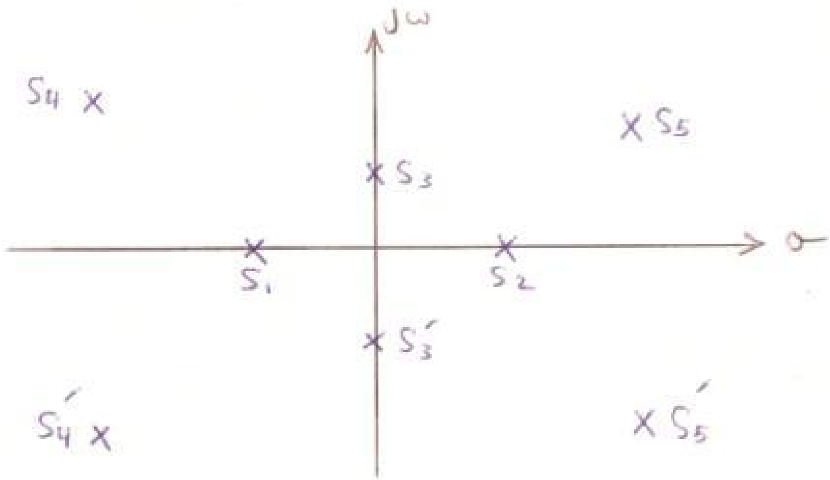
Then $s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$ is the C.E.

The roots of this equation are the poles of the T.F

Definition of stability

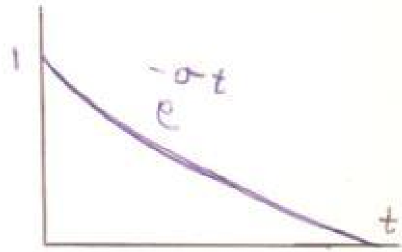
A system is said to be stable if for every bounded input, the output remains bounded.
Also, a system is stable if none of the poles of the closed-loop T.F may lie in the right hand half of the S-plane.

Ex. In the diagram shown below, explain with the aid of diagrams how the location of poles influences the system behaviour.



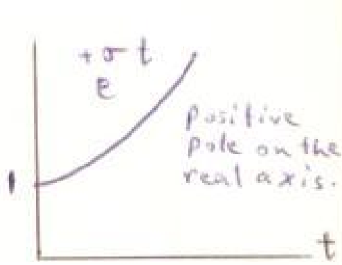
Ans. Using $S = \sigma + j\omega$

S_1 : implies a pole on the real axis where $\omega = 0$. This implies a simple exponential term in the solution.
Since S_1 is in the L.h.s of the S-plane
$\therefore$ σ must be negative.



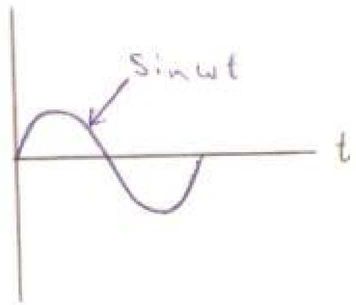
Therefore, the solution is a decaying exponential.

S_2 : is a pole on the real axis, but since it is in the r.h.s of the s-plane.
It therefore represents as exponential rice as σ is positive.

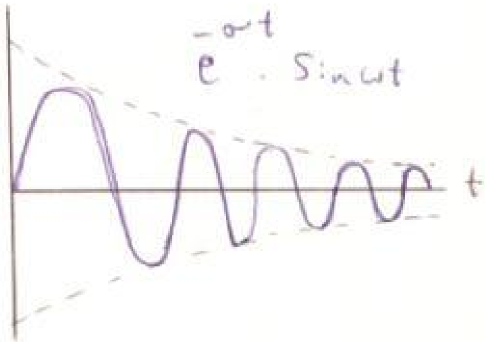


The poles S_3 and S_3' : imply an oscillatory term in the solution.

Because, they are on the $j\omega$ axis, there is no exponential term in the solution of these poles.

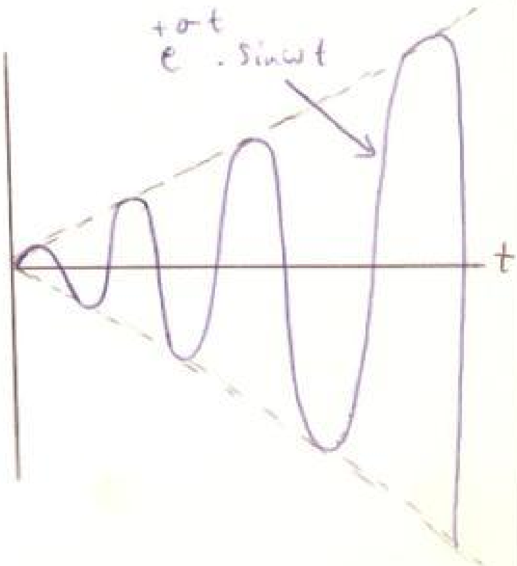


S_4 and S_4' : are Conjugate pair of poles off both axis.
Such poles on the L.h.s of the s-plane imply as oscillatory term which decays in magnitude exponentially.



The poles S_5 and S_5' :

are Conjugate pairs of poles off both axis, but in the r.h.s of the s-plane.
Therefore, the solution must be exponentially increasing oscillations.



The stability of Control system divided in to:

① - Absolute stability: It refers to the condition of stable or unstable, it is a yes or no condition, if the system has one positive pole (positive exponential) it is enough to say that the system is unstable. Another factor may effect the stability that is the amplifier gain (K), all the system has negative pole, i.e suppose to be stable but it meet be unstable if the amplifier gain increas over certain value called (K_{cr}).

② - Relative stability: Once the system is found tobe stable it is of interest to determine how stable it is and this degree of stability is a measure of relative stability.

Methods of determining stability:

1- Routh - Hurwitz Criterion.

2- Root Locus plot.

3- Nyquist Criterion.

4. Bode diagram.

5- Lyapunov's stability criterion.

Routh's stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them.

This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, iformation about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

①- Write the polynomial in (S) in the following form:

$$a_n S^n + a_{n-1} S^{n-1} + a_{n-2} S^{n-2} + \cdots + a_0 = 0$$

Where the coefficient are real quantities.

②- If any of the coefficient are zero or negative in the presence of at Least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable.

③- If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$\begin{array}{l|llll}
S^n & a_n & a_{n-2} & a_{n-4} & \cdots\; a_0 \\
S^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
S^{n-2} & b_1 & b_2 & b_3 & \cdots \\
S^{n-3} & c_1 & c_2 & c_3 & \cdots
\end{array}$$

$$b_1 = \frac{a_{n-1}\, a_{n-2} - a_n\, a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}\, a_{n-4} - a_n\, a_{n-5}}{a_{n-1}}$$

Then continue solve for b_3 and b_4 in the same way until obtain zeros.

$$C_1 = \frac{b_1\, a_{n-3} - b_2\, a_{n-1}}{b_1}$$

$$C_2 = \frac{b_1\, a_{n-5} - b_3\, a_{n-1}}{b_1}$$

Then continue to solve for C_3 and C_4 in the same way until obtain zeros after s^0.

If all the constants in the 1st column have the same sign (either +ve or −ve) then the system is stable. If the first column have +ve and −ve sign, then the system unstable. If one of these values equal to zero then the system is critical and the value of critical amplifier gain can be obtained accordingly.

<u>Ex1</u> Let us apply Routh's stability criterion to the following third order polynomial;

$$a_0 s^3 + a_1 s^2 + a_2 s + a_0 = 0$$

<u>Sol.</u>

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

$a_1 a_2 > a_3 a_0$ for the system to be stable.

Ex 2 Discuss stability using Routh criterion

$$S^3 - 2S^2 + 2S = 0$$

S^3	1	2	0
S^2	-2	0	0
S^1	2	0	0
S^0	0	0	0
	0	0	0

System unstable

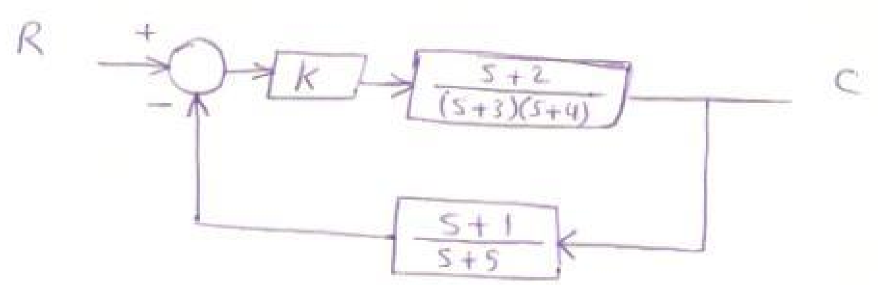
Ex 3 Discuss stability using Routh criterion

$$2S^4 + S^3 + 3S^2 + 5S + 10 = 0$$

S^4	2	3	10	0
S^3	1	5	0	0
S^2	-7	10	0	0
S^1	63	0	0	0
S^0	10	0	0	0
	0	0	0	0

System unstable

H.W 1 Find the value of K for the system to be stable.



H.W 2 Discuss stability equation

$$S^6 + 2S^5 + 8S^4 + 13S^3 + 20S^2 + 16S + 16 = 0$$

Root - Locus Analysis

Root Locus: It is the Locus of roots of the characteristic equation of the closed-Loop system as a specific parameter (usually, gain K) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-Loop pole or zero to the Locations of the closed-Loop poles.

Angle and Magnitude Conditions:

consider the transfer function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

The c/c equation is: $1 + G(s)\,H(s) = 0$

or $G(s)\,H(s) = -1$

Since $G(s)\,H(s)$ is complex quantity then

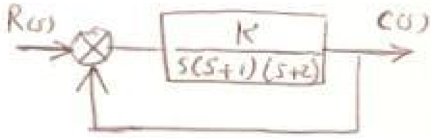
$G(s)\,H(s) = 1 \angle 180^\circ$

* Angle condition $\Rightarrow \angle G(s)\,H(s) = \mp 180(2k+1)$; $k = 0, 1, 2, \ldots$

* Magnitude Condition $\Rightarrow |G(s)\,H(s)| = 1$

A Locus of the points in the complex plane satisfying the angle condition alone is the root Locus. The roots of the c/c equation (the closed Loop poles) corresponding to a given value of the gain can be determined from the magnitude condition.

Ex: Consider the system shown in fig.



$$G(s) = \frac{K}{s(s+1)(s+2)}, \quad H(s) = 1$$

The angle condition is:

$$\angle G(s) = \angle \frac{K}{s(s+1)(s+2)} = -\angle s - \angle s+1 - \angle s+2$$

$$= \mp 180 (2n+1) \quad ; \quad (n = 0, 1, 2, \ldots)$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$$

Rules of Root Locus

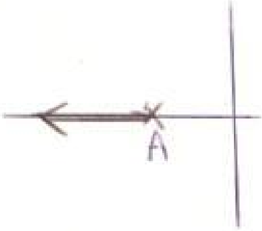
1. The number of Loci of the plot is equal to the order of the characteristic equation.
2. Each Locus start at an open Loop pole when $(K = 0)$ and finishes either an open Loop zero or infinite $(K = \infty)$
3. Locate the poles and zeros of $G(s)\,H(s)$ on the s-plane
4. Loci either moves along the real axis or occurs at the complex conjugate pairs of Loci (real axis acts a mirror)
5. Determine the asymptotes of root Loci.

$$\text{Angles of asymptotes} = \frac{\mp 180 (2n+1)}{P - Z} \quad (n = 0, 1, 2, \ldots)$$

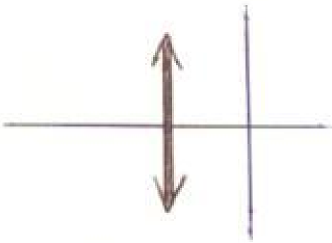
Where

P = number of poles of $G(s)\,H(s)$.

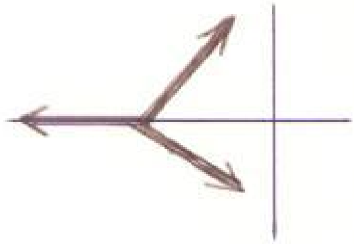
Z = number of zeros of $G(s)\,H(s)$.



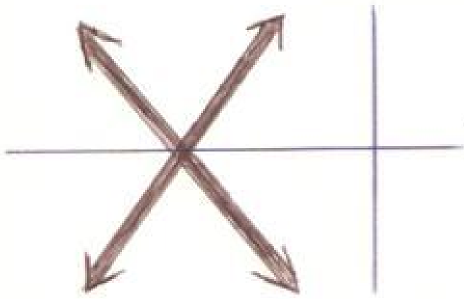
1 Asymptot



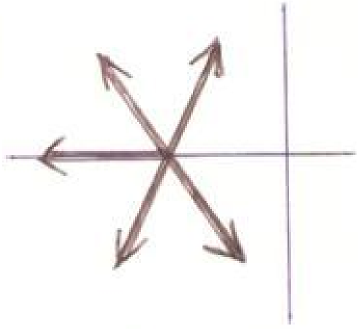
2 Asymptot



3 Asymptot



4 Asymptot



5 Asymptot

⑥- The number of Asymptot in the root Locus equal to $(P - Z)$

⑦- Those asymptotics intersect the real axis in one point (A) where

$$A = \frac{\sum \text{poles} - \sum \text{zeros}}{P - Z}$$

⑧. Find the breakaway and break-in points. Because of the Conjugate symmetry of the root Loci, the breakaway points and break-in points either Lie on the real axis or occur in Complex-conjugate pairs.

If a root Locus Lies between two adjacent open-loop Poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros.

Suppose that the c/c equation is given by

$A(s) + K B(s) = 0$

The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{A'(s)\,B(s) - A(s)\,B'(s)}{B^2(s)}$$

If the value of K corresponding to root $s = s_1$ of $\frac{dK}{ds} = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. If the value of K is negative, then point $s = s_1$ is not a breakaway or break-in point.

⑦- Find critical value of K and the points where the root loci cross the imaginary axis using Routh criterion or by letting $s = j\omega$ in the c/c equation, equating both the real part and the imaginary part to zero, and solving for ω and K.

Ex. Find the critical value of K and points of ~~intersect~~ intersection between root Loci and imaginary axis for c/c equation below :-

$$s^3 + 3s^2 + 2s + K = 0$$

1st method:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & 0 \\ s^0 & K & 0 \end{array}$$

$\therefore$ critical $K = 6$

Solving the auxiliary equation obtained from the s^2 row.

$$3s^2 + K = 3s^2 + 6 = 0 \Rightarrow s^2 = -2 \Rightarrow s = \mp j\sqrt{2}$$

2nd method:

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

$$K - 3\omega^2 = 0$$

$$2\omega - \omega^3 = 0$$

for which $\omega = \mp\sqrt{2}$, $K = 6$

10 - The angles of departure (or angles of arrival) of root Loci from the complex poles (or at complex zeros) is given by :

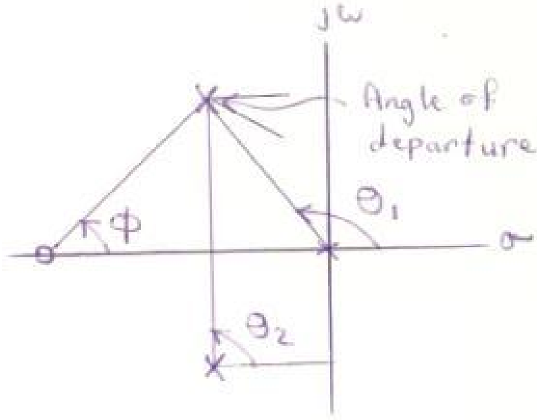
$$\theta_d = 180^\circ - \theta_p + \Phi_z$$

Where

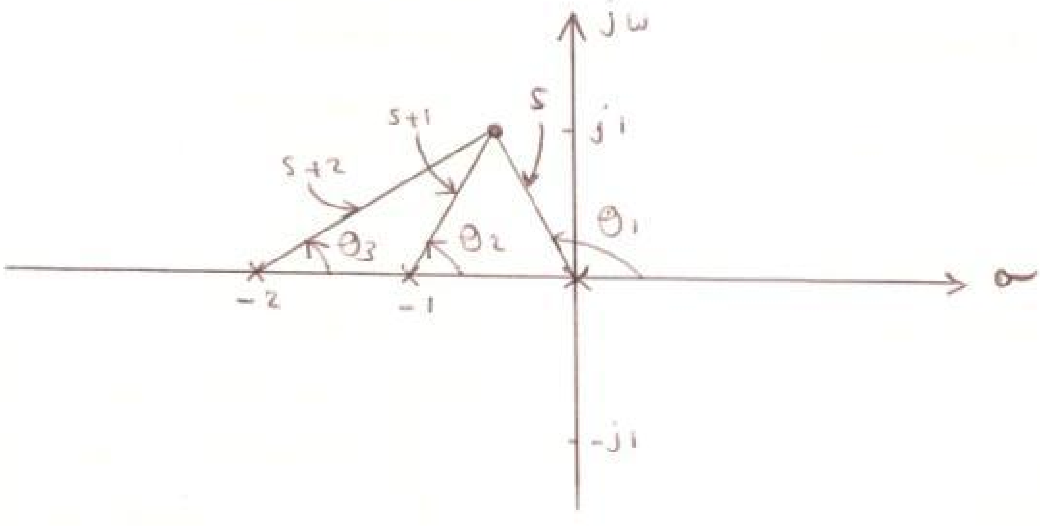
θ_d = angle of departure

θ_p = is the sum of the all angles subtended by other poles.

Φ_z = is the sum of the all angles subtended by other zeros.



⑪. Choose a test point in the broad neighborhood of the jω axis and the origin, as shown in fig. below, and apply the angle condition. If the test point is on the root loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must by 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition.



<u>Ex.</u> $G(s) = \dfrac{K}{s(s+1)(s+2)}$, $H(s) = 1$, (K is non negative)

sketch the root-Locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant Complex-Conjugate closed-Loop poles is 0.5.

<u>Solution:-</u>

①. No. of Loci = 3

②. No. of asymptotes = 3

Angles of asymptotes $= \dfrac{\mp 180\,(2n+1)}{P-Z} = \dfrac{\mp 180\,(2n+1)}{3}$

$= \mp 60\,(2n+1)$; $(n = 0, 1, 2, \dots)$

$= 60^\circ , -60^\circ , 180^\circ$

$A = \dfrac{\sum \text{poles} - \sum \text{zeros}}{P-Z} = \dfrac{0+(-1)+(-2)}{3} = \dfrac{-3}{3} = -1$

③- The c/c equation is

$G(s) + 1 = 0 \implies \dfrac{K}{s(s+1)(s+2)} + 1 = 0$

$K = -(s^3 + 3s^2 + 2s)$

$\dfrac{dK}{ds} = 0 \implies \dfrac{dK}{ds} = -(3s^2 + 6s + 2) = 0$

$s = -0.4226$; $s = -1.5774$

$s = -0.4226$ is corresponds to the actual breakaway point.

$K = 0.3849$ for $s = -0.4226$

$K = -0.3849$ for $s = -1.5774$

④. The points where the root loci cross the imaginary axis are:

$$K = 6 \quad ; \quad s = \mp j\sqrt{2}$$

⑤. $\zeta = 0.5$; $\mp \cos^{-1}\zeta = \mp \cos^{-1} 0.5 = \mp 60^\circ$

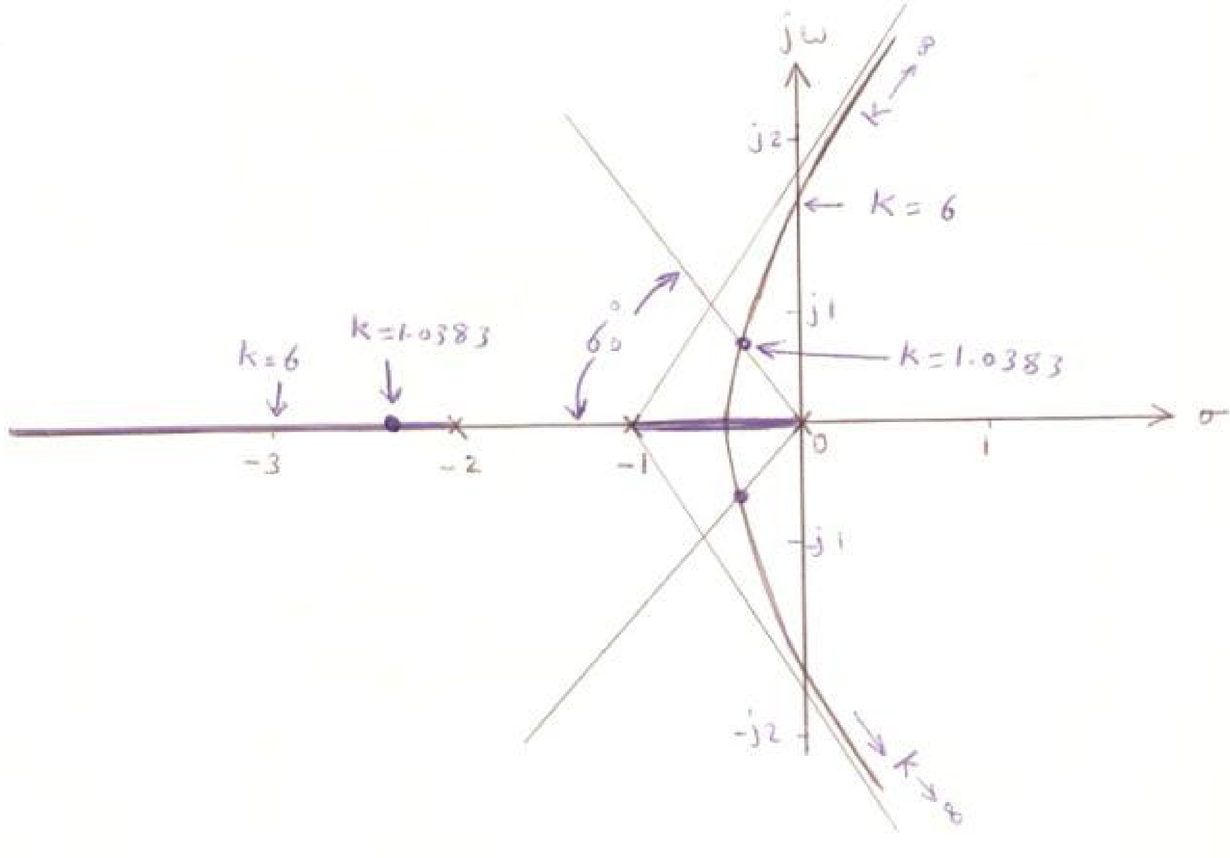
$s_1 = -0.3337 + j0.578$; $s_2 = -0.3337 - j0.578$

From magnitude condition

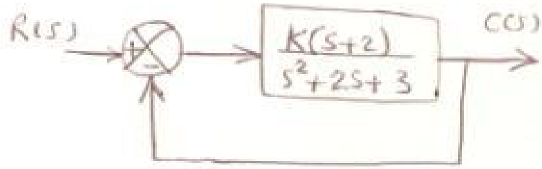
$$K = \left| s(s+1)(s+2) \right|_{s = -0.3337 + j0.578}$$

$$K = 1.0383$$

Using this value of K, the third pole is found at $s = -2.3326$



Ex2: For the system shown in figure, sketch the root-Locus plot.

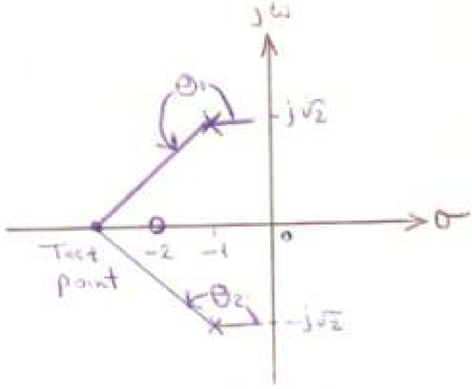


Solution: $G(s) = \frac{K(s+2)}{s^2+2s+3}$; $H(s) = 1$ where $K \geq 0$.

G(s) has a pair of complex conjugate poles at

$s = -1 + j\sqrt{2}$; $s = -1 - j\sqrt{2}$

The angle of departure from the complex-conjugate open-Loop poles.



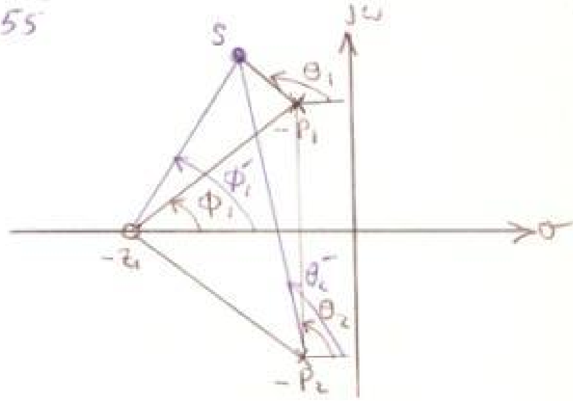
$$\Phi_1' - (\theta_1 + \theta_2') = \mp 180(2n+1)$$

or

$$\theta_1 = 180 - \theta_2' + \Phi_1' = 180 - \theta_2 + \Phi_1$$

$$\theta_1 = 180 - \theta_2 + \Phi_1 = 180^\circ - 90^\circ + 55^\circ$$

$$\theta_1 = 145^\circ$$



Determine the break-in point

$$K = -\frac{s^2+2s+3}{s+2}$$

$$\frac{dk}{ds} = -\frac{(2s+2)(s+2) - (s^2+2s+3)}{(s+2)^2} = 0$$

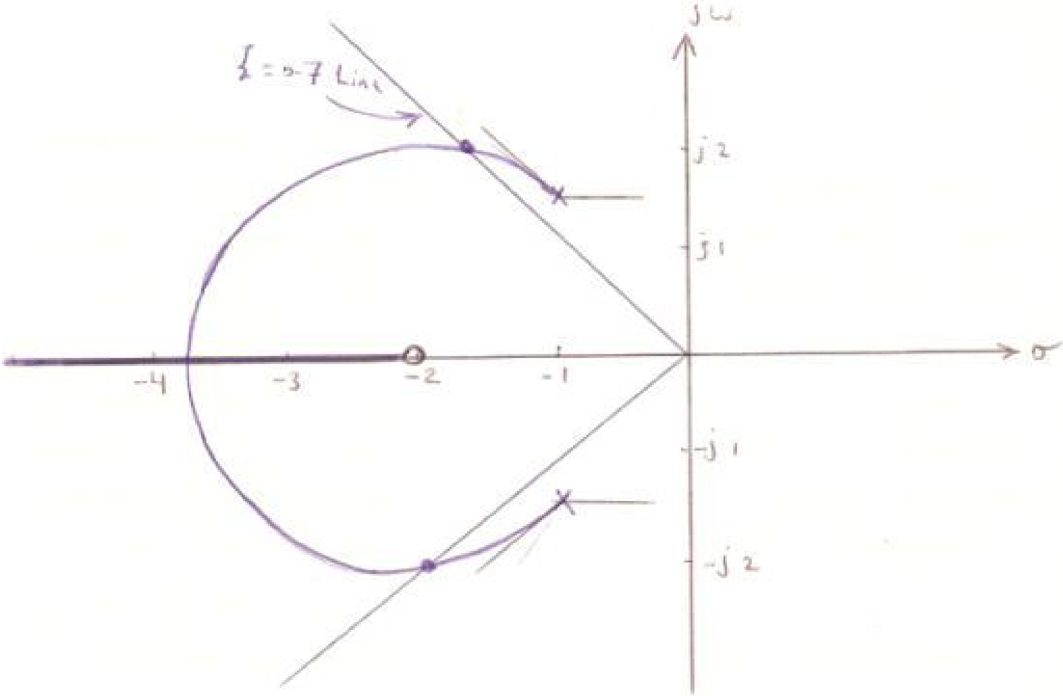
which gives $s^2 + 4s + 1 = 0$

or $s_1 = -3.7320$ or $s_2 = -0.2680$

hence $s = -3.7320$ is break-in point

and $K = 5.4641$ at this point.

(Note that at $s = -0.2680$; $K = -1.4641$)



The value of the gain K at any point on the root Locus can be found by applying the magnitude condition.

at $\zeta = 0.7$

$$K = \left| \frac{(s+1-j\sqrt{2})(s+1+j\sqrt{2})}{s+2} \right|_{s=-1.67+j1.7}$$

$$= 1.34$$

To show the occurrence of a circular root-Locus, we need to derive the equation for the root locus.

For this system, the angle condition is

$$\angle s+2 - \angle s+1-j\sqrt{2} - \angle s+1+j\sqrt{2} = \mp 180(2n+1)$$

if $s = \sigma + j\omega$, we obtain

$$\angle \sigma+2+j\omega - \angle \sigma+1+j\omega-j\sqrt{2} - \angle \sigma+1+j\omega+j\sqrt{2} = \mp 180(2n+1)$$

Which can be written as

$$\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) - \tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = \mp 180(2n+1)$$

or

$$\tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right) + \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \mp 180^\circ(2n+1)$$

Taking tangents of both sides of this Last equation using the relationship $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

We obtain

$$\frac{\frac{\omega-\sqrt{2}}{\sigma+1} + \frac{\omega+\sqrt{2}}{\sigma+1}}{1 - \left(\frac{\omega-\sqrt{2}}{\sigma+1}\right)\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right)} = \frac{\frac{\omega}{\sigma+2} \pm 0}{1 \mp \left(\frac{\omega}{\sigma+2}\right)(0)}$$

Which can be simplified as $\dfrac{2\omega(\sigma+1)}{(\sigma+1)^2 - (\omega^2-2)} = \dfrac{\omega}{\sigma+2}$

or $\omega\left[(\sigma+2)^2 + \omega^2 - 3\right] = 0$

This Last equation is equivalent to

$$\omega = 0 \quad \text{or} \quad (\sigma+2)^2 + \omega^2 = (\sqrt{3})^2$$

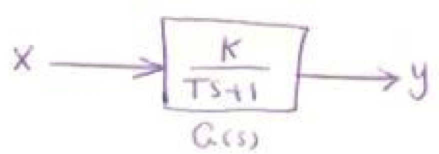
first equation $\omega = 0$, the real axis from $s = -2$ to $s = -\infty$

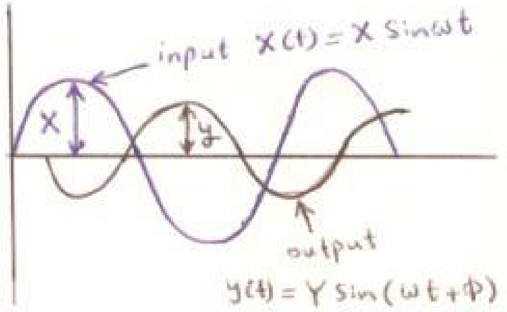
Second equation $(\sigma+2)^2 + \omega^2 = (\sqrt{3})^2$, the root Locus is a circular has radius equal $\sqrt{3}$ and Center at $\sigma = -2$ and $\omega = 0$, Where $K \geq 0$ (positive value).

Frequency - Response Analysis

The term frequency response means the steady-state response of a system to a sinusoidal input, i.e, we vary the frequency of the input signal over a certain range and study the resulting frequency response.

Consider the following system





If $x(t) = X \sin\omega t$ then $y(t)$ can be found as follows:
substituting $j\omega$ for s in $G(s)$ yields

$$G(j\omega) = \frac{K}{j\omega T + 1}$$

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$

while the phase angle ϕ is

$$\phi = \angle G(j\omega) = -\tan^{-1}\omega T$$

Thus $$y(t) = \frac{XK}{\sqrt{1+T^2\omega^2}} \sin(\omega t - \tan^{-1}\omega T)$$

This system is a phase-Lag system for Large ω.

Ex:- Consider the network given by

$$G(s) = \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}}$$

Determine whether the network is a Lead or Lag network.

Solution: $x(t) = X \sin \omega t$

$$G(j\omega) = \frac{j\omega + \frac{1}{T_1}}{j\omega + \frac{1}{T_2}} = \frac{T_2 (1 + T_1 j\omega)}{T_1 (1 + T_2 j\omega)}$$

we have

$$|G(j\omega)| = \frac{T_2 \sqrt{1 + T_1^2 \omega^2}}{T_1 \sqrt{1 + T_2^2 \omega^2}}$$

and

$$\phi = \underline{/G(j\omega)} = \tan^{-1} T_1 \omega - \tan^{-1} T_2 \omega$$

$$y(t) = \frac{X T_2 \sqrt{1 + T_1^2 \omega^2}}{T_1 \sqrt{1 + T_2^2 \omega^2}} \sin(\omega t + \tan^{-1} T_1 \omega - \tan^{-1} T_2 \omega)$$

if $T_1 > T_2$, then $\tan^{-1} T_1 \omega - \tan^{-1} T_2 \omega > 0$

Thus if $T_1 > T_2$

then the network is a Lead network.

If $T_1 < T_2$

then the network is a Lag network.

Frequency-Response characteristics in graphical Forms

1. Nyquist plot or polar plot.
2. Bode diagram or Logarithmic plot.
3. Log-magnitude-Versus phase plot (Nichols plots).

polar plot (Nyquist plot)

It is a plot of magnitude of $G(j\omega)$ Varies with the Phase angle of $G(j\omega)$ on a polar Coordinate as ω Varied from $0 \rightarrow \infty$. It is often called the Nyquist plot.

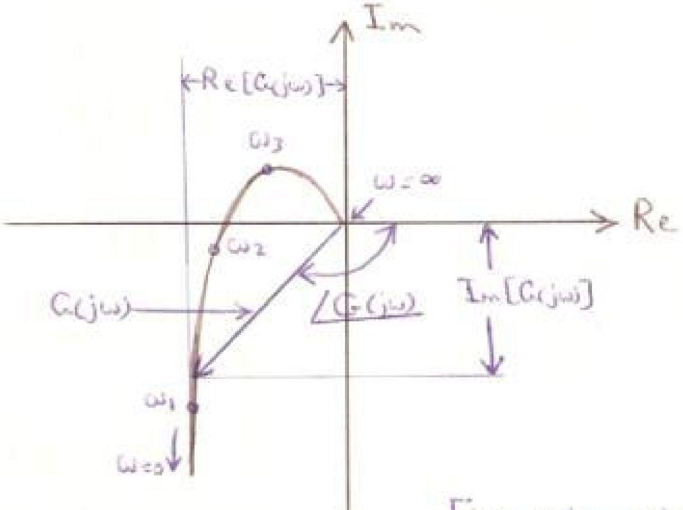


Fig. polar plot

Information about stability is available directly from a polar plot of the sine response of open Loop transfer function $G(j\omega)H(j\omega)$.

* Integral and derivative factors $(j\omega)^{\mp 1}$

$$G(j\omega) = \frac{1}{j\omega} \quad \text{is the negative imaginary axis.}$$

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega} \angle -90^\circ$$

The polar plot of $G(j\omega) = j\omega$ is the positive imaginary axis.

* First-order factors $(1+j\omega T)^{\mp 1}$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

$$G(j0) = 1\angle 0^\circ \quad \text{at } \omega = 0$$

$$G(j\tfrac{1}{T}) = \frac{1}{\sqrt{2}} \angle -45^\circ \quad \text{at } \omega = \frac{1}{T}$$

$$G(j\omega) = X + jY$$

where $X = \dfrac{1}{1+\omega^2T^2}$ = real part of $G(j\omega)$.

$Y = \dfrac{-\omega T}{1+\omega^2T^2}$ = imaginary part of $G(j\omega)$.

Then we obtain

$$(X-\tfrac{1}{2})^2 + Y^2 = \left(\frac{1}{2}\,\frac{1-\omega^2T^2}{1+\omega^2T^2}\right)^2 + \left(\frac{-\omega T}{1+\omega^2T^2}\right)^2 = \left(\frac{1}{2}\right)^2$$

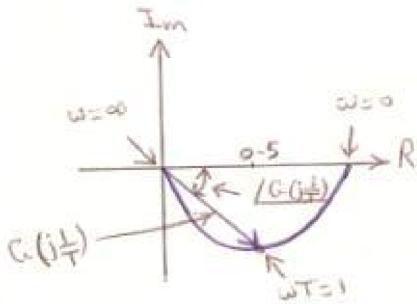


Fig. Polar plot of $\dfrac{1}{1+j\omega T}$

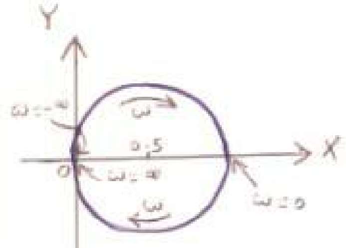


Fig. plot of $G(j\omega)$ in X-Y plane

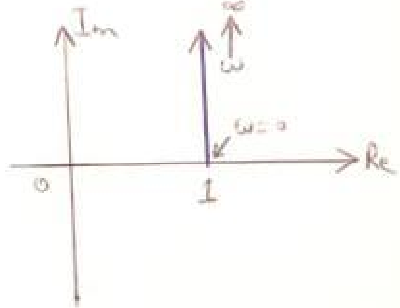


Fig. polar plot of $1+j\omega T$

* Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\mp 1}$

$$G(j\omega)=\frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2} \quad \text{for } \zeta>0$$

$$\lim_{\omega\to 0} G(j\omega)=1\angle 0^\circ \;;\quad \lim_{\omega\to\infty} G(j\omega)=0\angle 180^\circ$$

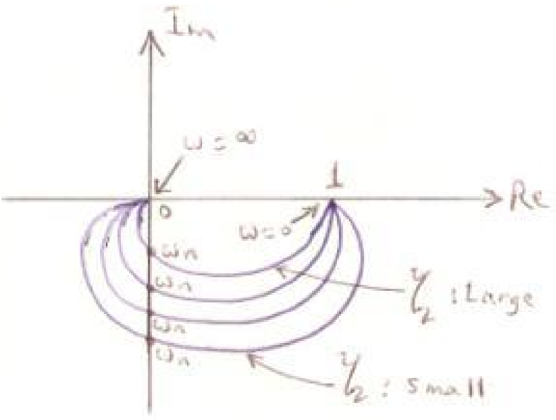


Fig. polar plot of $\frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2}$; for $\zeta>0$

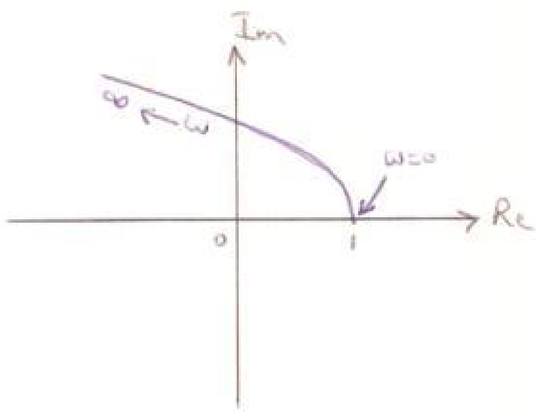


Fig. polar plot of $1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2$; for $\zeta>0$

For $G(j\omega)=1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2$

$$=(1-\frac{\omega^2}{\omega_n^2})+j(\frac{2\zeta\omega}{\omega_n})$$

The low-frequency portion of the curve is

$$\lim_{\omega\to 0} G(j\omega)=1\angle 0^\circ$$

and the high-frequency is

$$\lim_{\omega\to\infty} G(j\omega)=\infty\angle 180^\circ$$

Ex 1: Sketch a polar plot of the T.F

$$G(s) = \frac{1}{s(Ts+1)}$$

Solution: $G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$

$$= -\frac{T}{1+\omega^2T^2} - j\frac{1}{\omega(1+\omega^2T^2)}$$

$$\lim_{\omega\to 0} G(j\omega) = -T - j\infty = \infty \angle -90^\circ$$

$$\lim_{\omega\to\infty} G(j\omega) = 0 - j0 = 0 \angle -180^\circ$$

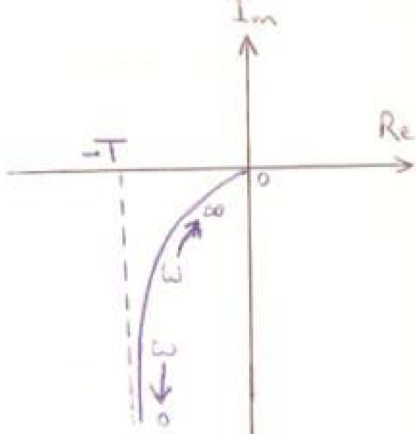


Fig. polar plot of $\frac{1}{j\omega(1+j\omega T)}$

Ex 2: Sketch a polar plot of the T.F

$$G(j\omega) = \frac{e^{-j\omega L}}{1+j\omega T}$$

Solution: $G(j\omega) = (e^{-j\omega L})\left(\frac{1}{1+j\omega T}\right)$

$$|G(j\omega)| = |e^{-j\omega L}| \cdot \left|\frac{1}{1+j\omega T}\right|$$

$$= \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$\angle G(j\omega) = \angle e^{-j\omega L} + \angle \frac{1}{1+j\omega T}$$

$$= -\omega L - \tan^{-1}\omega T$$

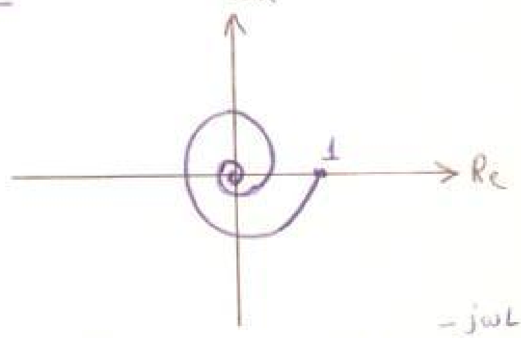
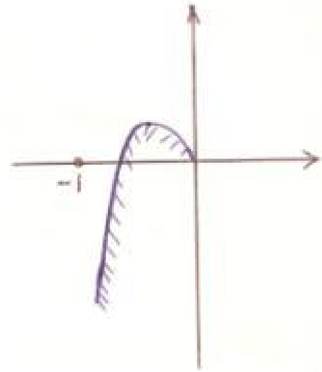


Fig. polar plot of $\frac{e^{-j\omega L}}{1+j\omega T}$

Nyquist Stability Criterion



If the function has No. of Zeros and poles in the right half of the s-plane, then for closed Loop system to be stable the Nyquist plot must not enclosed the critical point $(-1, j0)$.

The Nyquist stability criterion determines the stability of a closed-Loop system from its open-Loop frequency response and open-Loop poles.

Stability Analysis

In examining the stability of Linear control systems using the Nyquist stability criterion, we see that three possibilities can occur:

1. There is no encirclement of the $(-1+j0)$ point. This implies that the system is stable if there are no poles of $G(s)\,H(s)$ in the right-half s plane; otherwise, the system is unstable.
2. There are one or more counterclockwise encirclements of the $(-1+j0)$ point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of $G(s)\,H(s)$ in the right-half s plane; otherwise, the system is unstable.
3. There are one or more clockwise encirclements of the $(-1+j0)$ point. In this case the system is unstable.

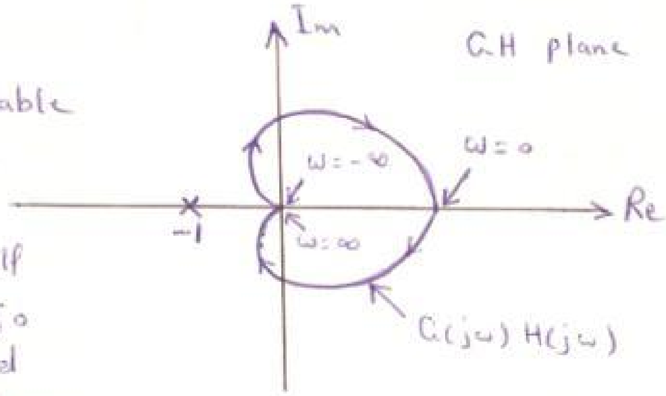
Ex1 Consider a closed-Loop system whose open-Loop transfer function is given by

$$G(s)\,H(s) = \frac{K}{(T_1 s+1)(T_2 s+1)}$$

Examine the stability of the system.

Solution:

The system is stable because of $G(s)\,H(s)$ does not have any poles in the right-half s plane and the $-1+j0$ point is not encircled by the $G(j\omega)\,H(j\omega)$ Locus.



Ex2: Consider the system with the following open-Loop transfer function

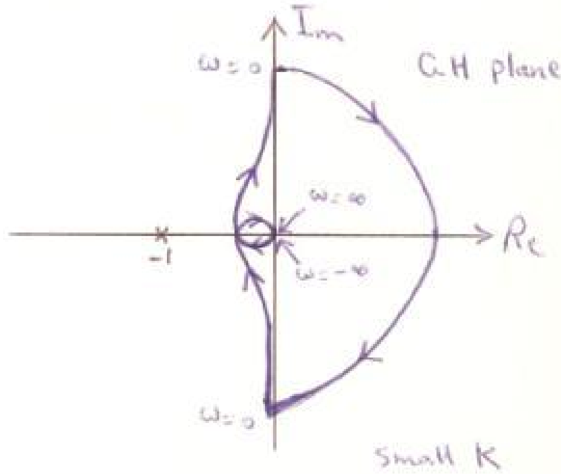
$$G(s)\,H(s) = \frac{K}{s(T_1 s+1)(T_2 s+1)}$$

Determine the stability of the system for two cases (1) the gain K is small ; (2) K is Large

Solution:

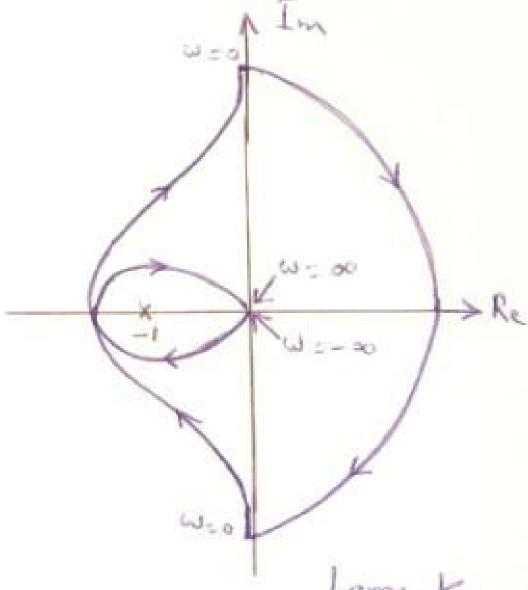
(1) for K is small

The system is stable because of the $G(s)\,H(s)$ Locus not encircle the $(-1+j0)$ point.



(2) for Large K

The Locus of $G(s)\,H(s)$ encircles the $(-1+j0)$ point twice in the clockwise direction, indicating two closed-Loop poles in the right-half s-plane, and the system is unstable.



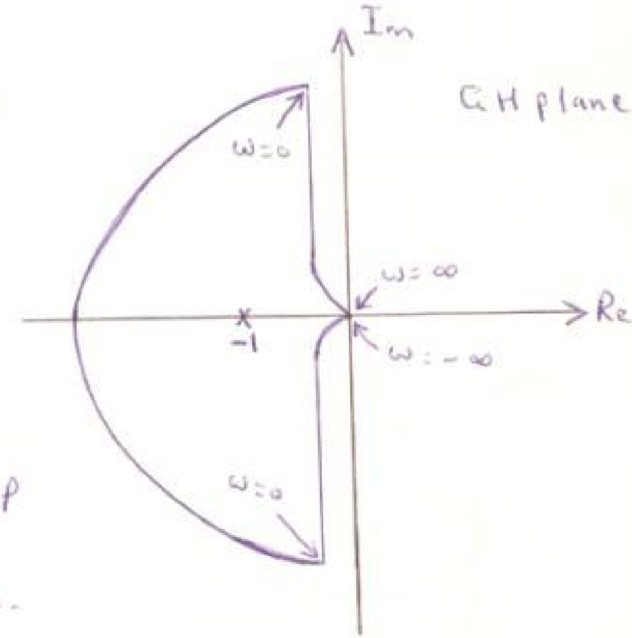
Large K

Ex3: Consider the closed-Loop system having the following open-Loop transfer function:

$$G(s)\,H(s) = \frac{K}{s(Ts-1)}$$

Determine the stability of the system.

Solution:



The $G(s)\,H(s)$ plot encircles the $(-1+j0)$ point once clockwise and has one pole ($s=\frac{1}{T}$) in the right-half s plane.

Therefore, this means that the closed-Loop system has two closed-Loop poles in the right-half s plane and is unstable.

Bode Diagrams or Logarithmic plots

A Bode diagram consists of two graphs:
One is a plot of the Logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a Logarithmic scale.

The standard representation of the Logarithmic magnitude of $G(j\omega)$ is $20 \, \text{Log} |G(j\omega)|$, where the base of the Logarithm is 10. The unit used in this representation of the magnitude is the decibel (dB), while the phase angle (in degree).

Basic Factors of $G(j\omega) H(j\omega)$

1- Gain K.
2- Integral and derivative factors $(j\omega)^{\mp 1}$.
3- First-order factors $(1+j\omega T)^{\mp 1}$.
4- Quadratic factors $\left[1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]^{\mp 1}$.

* The gain K

if $K>1$ then $\text{Log}\, K$ is +ve.
if $K<1$ then $\text{Log}\, K$ is −ve.

The Log-magnitude curve for a constant gain K is a horizontal straight Line at the magnitude of $20 \, \text{Log}\, K$ decibels. The phase angle of the gain K is zero.

Also,

$$20 \log (K \times 10) = 20 \log K + 20$$

Similarly, $$20 \log (K \times 10^n) = 20 \log K + 20n$$

Note:

$$20 \log K = -20 \log \frac{1}{K}$$

* Integral and Derivative Factors $(j\omega)^{\mp 1}$

The Logarithmic magnitude of $1/j\omega$ in dB is

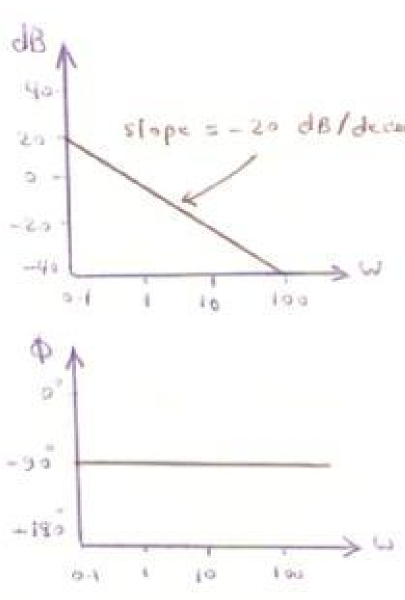
$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \ \text{dB}.$$

The phase angle of $1/j\omega$ is constant and equal to -90° or $-\frac{\pi}{2}$.

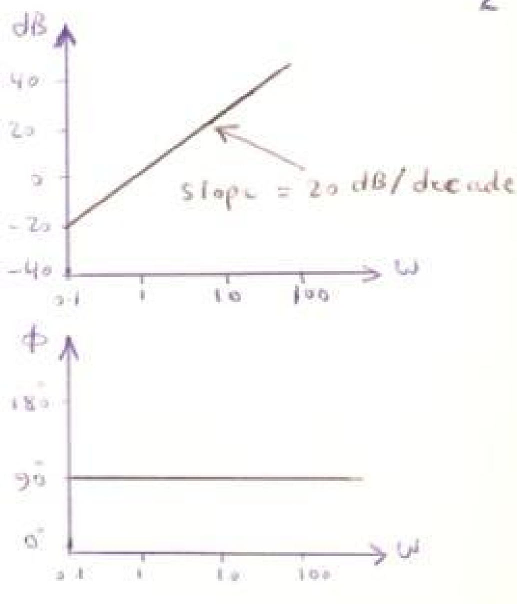
Similarly, the log-magnitude of $j\omega$ in dB is

$$20 \log (j\omega) = 20 \log \omega \ \text{dB}.$$

The phase angle of $j\omega$ is constant and equal to 90° or $\frac{\pi}{2}$.



(a) For $G(j\omega) = \frac{1}{j\omega}$



(b) For $G(j\omega) = j\omega$

If the transfer function contains the factor $(1/j\omega)^n$ or $(j\omega)^n$, the Log magnitude becomes, respectively,

$$20 \operatorname{Log} \left| \frac{1}{(j\omega)^n} \right| = -n \times 20 \operatorname{Log} |j\omega| = -20n \operatorname{Log} \omega \quad \text{dB}$$

or

$$20 \operatorname{Log} |(j\omega)^n| = n \times 20 \operatorname{Log} |j\omega| = 20n \operatorname{Log} \omega \quad \text{dB}$$

The phase angle $= -90^\circ \times n$ for $(1/j\omega)^n$

and equal to $90^\circ \times n$ for $(j\omega)^n$.

* First-order Factors $(1+j\omega T)^{\mp 1}$

The Log magnitude of the first-order factor $1/(1+j\omega T)$ is

$$20 \operatorname{Log} \left| \frac{1}{1+j\omega T} \right| = -20 \operatorname{Log} \sqrt{1+\omega^2 T^2} \quad \text{dB}$$

For Low frequencies ($\omega \leqslant \frac{1}{T}$):

$$-20 \log \sqrt{1+\omega^2 T^2} \doteqdot -20 \operatorname{Log} 1 = 0 \, \text{dB}$$

For high frequencies ($\omega \gg \frac{1}{T}$):

$$-20 \operatorname{Log} \sqrt{1+\omega^2 T^2} \doteqdot -20 \operatorname{Log} \omega T \quad \text{dB}$$

at $\omega = \frac{1}{T} \Rightarrow$ Log magnitude $= 0$ dB

at $\omega = \frac{10}{T} \Rightarrow$ Log magnitude $= -20$ dB

The phase angle $= \phi = -\tan^{-1} \omega T$

at $\omega = 0 \Rightarrow \phi = 0$

at $\omega = \frac{1}{T} \Rightarrow \phi = -45^\circ$

at $\omega = \infty \Rightarrow \phi = -90^\circ$

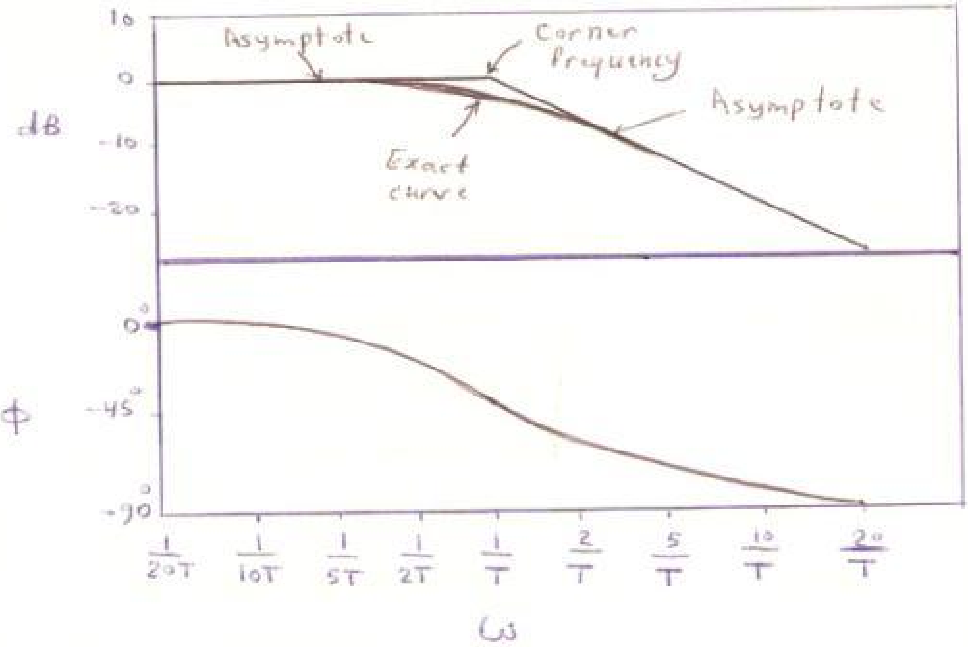


Fig. For factor $1/(1+j\omega T)$ [magnitude and phase angle]

For $(1+j\omega T)$, the Log magnitude and the phase angle curves need only be changed in sign with respect to the factor $[1/(1+j\omega T)]$; since

$$20 \log(1+j\omega T) = -20 \log \left| \frac{1}{1+j\omega T} \right|$$

and $\angle 1+j\omega T = \tan^{-1}\omega T = -\angle \frac{1}{1+j\omega T}$

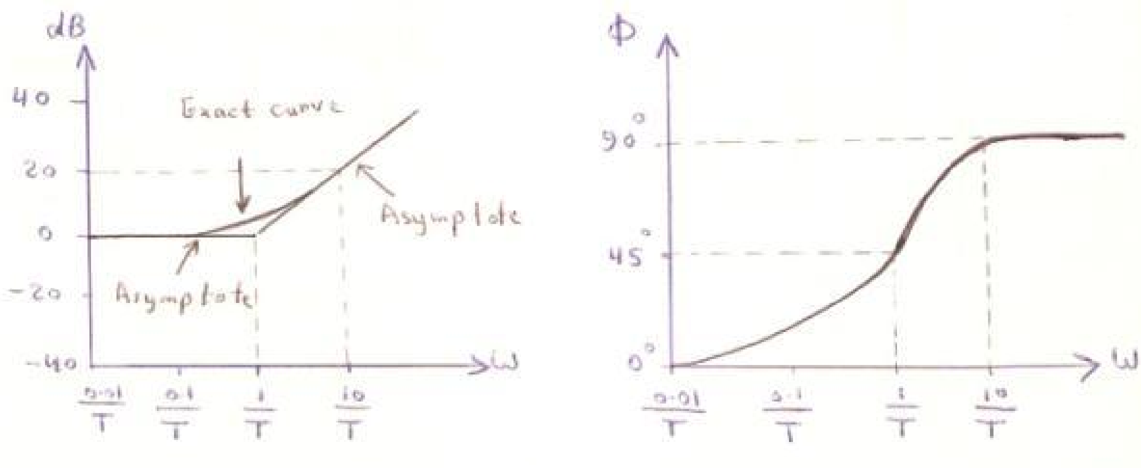


Fig. For factor $(1+j\omega T)$ [magnitude and phase angle]

* Quadratic Factors $\left[1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]^{\mp 1}$

$$G(j\omega) = \frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}$$

$$20\log\left|\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\,\mathrm{Log}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

For Low frequencies ($\omega \ll \omega_n$), the Log magnitude becomes

$$-20\,\mathrm{Log}\,1 = 0\,dB$$

For high frequency ($\omega \gg \omega_n$), the Log magnitude becomes

$$-20\,\mathrm{Log}\frac{\omega^2}{\omega_n^2} = -40\,\mathrm{Log}\frac{\omega}{\omega_n}\ dB$$

The high-frequency asymptote equation is a straight Line having the slope -40 dB/decade since

$$-40\log\frac{10\omega}{\omega_n} = -40-40\,\mathrm{Log}\frac{\omega}{\omega_n}$$

The high-frequency asymptote intersects the Low-frequency one at $\omega=\omega_n$ since at that frequency

$$-40\,\mathrm{Log}\frac{\omega_n}{\omega_n} = -40\,\mathrm{Log}\,1 = 0\,dB$$

This frequency, ω_n, is the corner frequency for the quadratic factor considered.

The phase angle of the quadratic factor $\left[1+\left(j\frac{\omega}{\omega_n}\right)2\zeta+\left(j\frac{\omega}{\omega_n}\right)^2\right]^{-1}$ is

$$\phi = \angle\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

The phase angle is a function of both ω and ζ.

At $\omega = 0 \Rightarrow \phi = 0$

At corner frequency $\omega = \omega_n \Rightarrow \phi = -90^\circ$

At $\omega = \infty \Rightarrow \phi = -180^\circ$

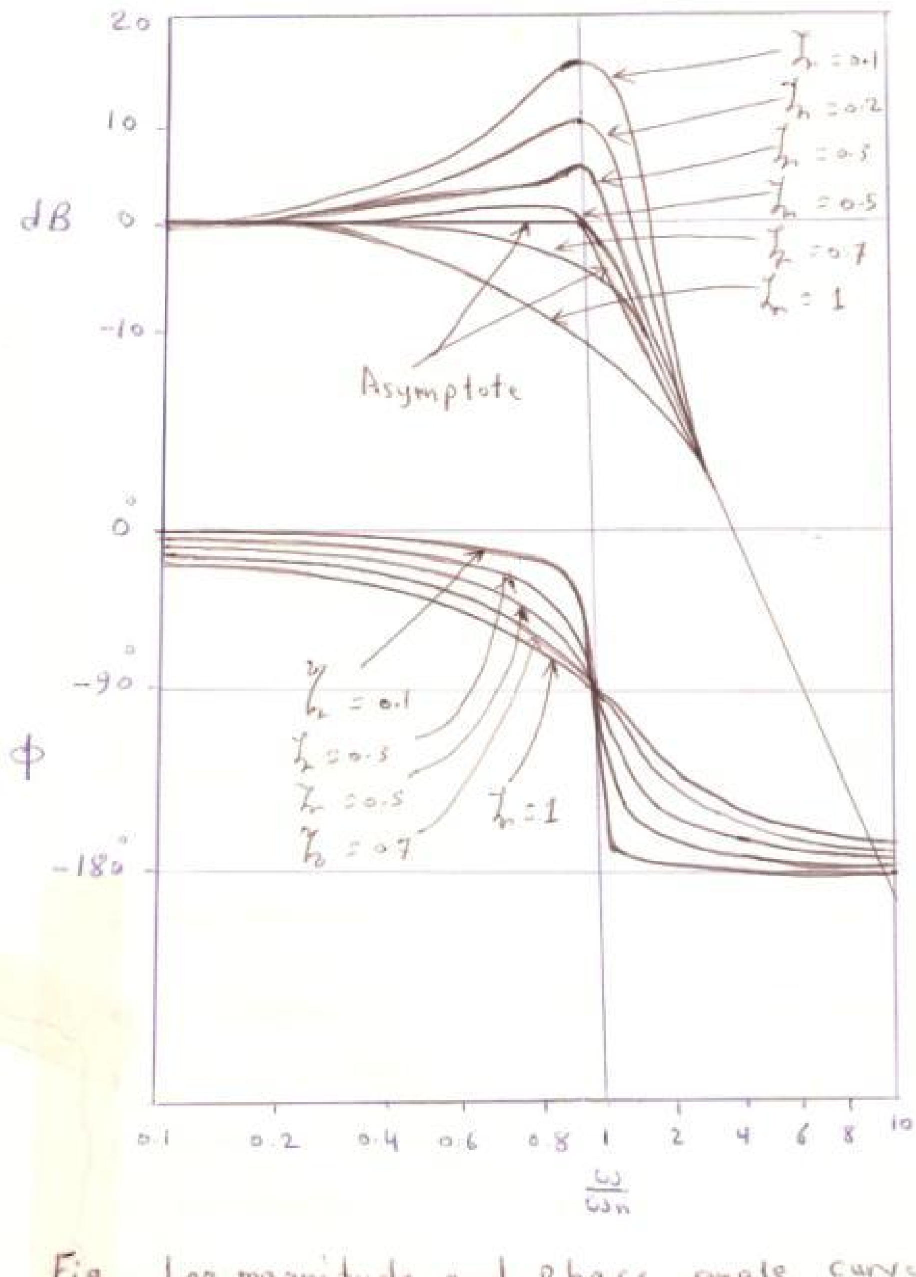


Fig. Log magnitude and phase angle curves of the quadratic transfer function $G(j\omega) = \dfrac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$

Ex1: Draw the Bode diagram for the following transfer function.

$$G(j\omega) = \frac{10\,(j\omega+3)}{(j\omega)(j\omega+2)\left[(j\omega)^2+j\omega+2\right]}$$

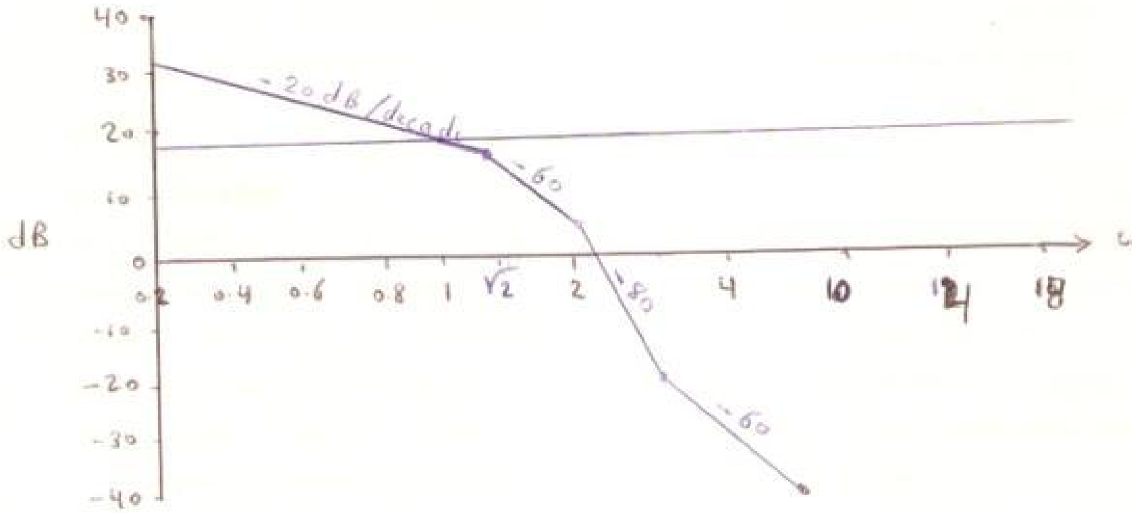
Solution:

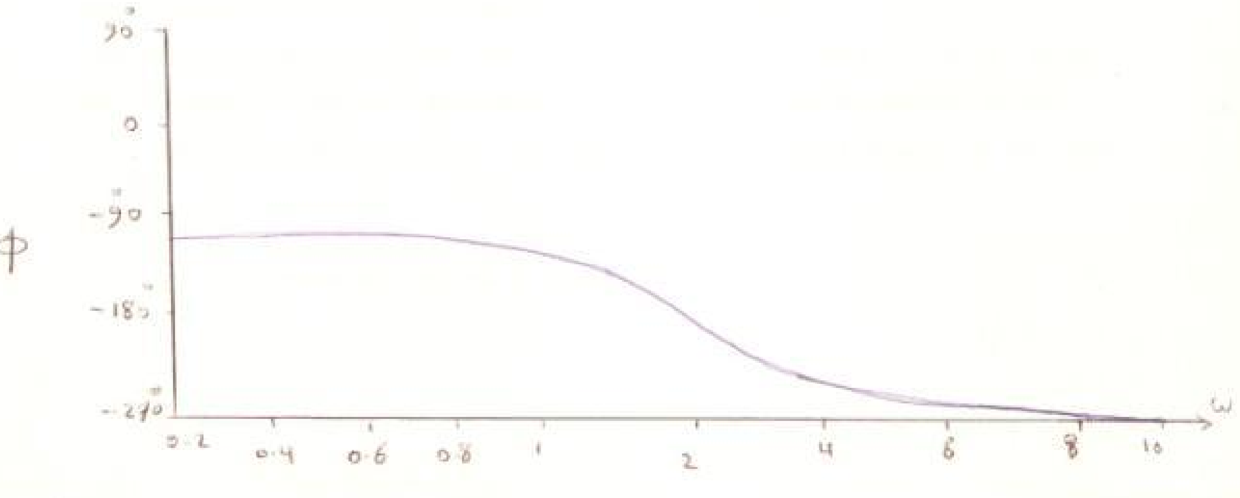
$$G(j\omega) = \frac{7.5\left(\frac{j\omega}{3}+1\right)}{(j\omega)\left(\frac{j\omega}{2}+1\right)\left[\frac{(j\omega)^2}{2}+\frac{j\omega}{2}+1\right]}$$

This function is composed of the following factors:

$7.5 \quad (j\omega)^{-1} \quad 1+j\frac{\omega}{3} \quad \left(1+j\frac{\omega}{2}\right)^{-1} \quad \left[1+j\frac{\omega}{2}+\frac{(j\omega)^2}{2}\right]^{-1}$

The corner frequencies of the third, fourth, and fifth terms are $\omega = 3$; $\omega = 2$, and $\omega = \sqrt{2}$ respectively.





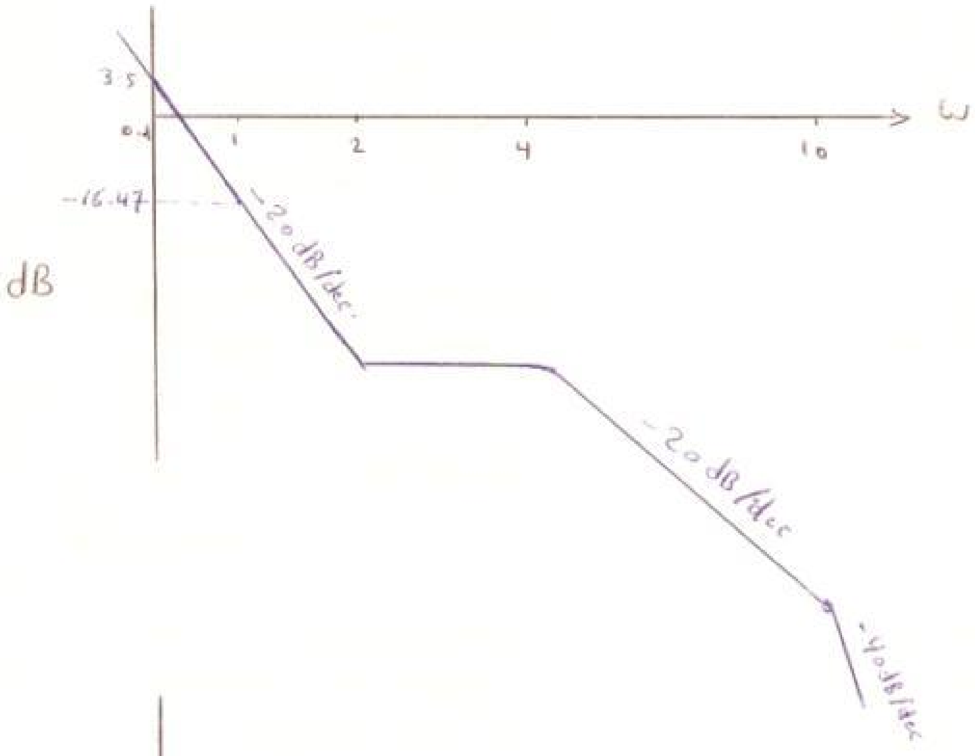
Ex2: Draw the Bode diagram for the following transfer function:

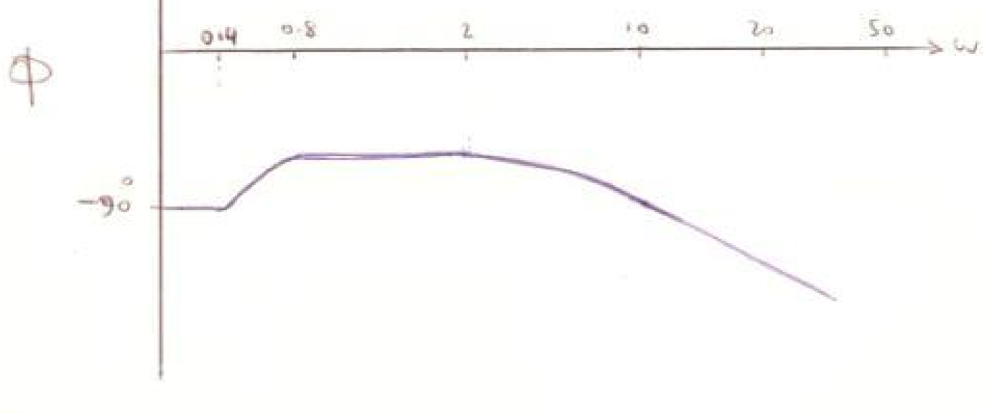
$$GH(s) = \frac{3(s+2)}{s(s+4)(s+10)}$$

Solution:

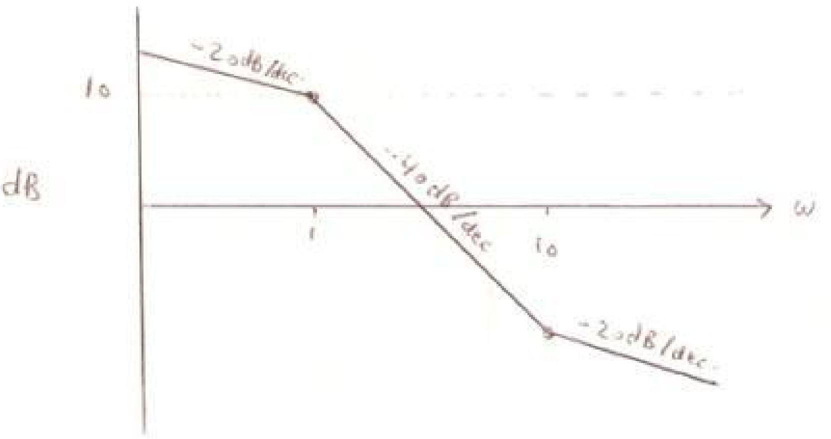
$$GH(j\omega) = \frac{6/40 \left(\frac{j\omega}{2}+1\right)}{(j\omega)\left(\frac{j\omega}{4}+1\right)\left(\frac{j\omega}{10}+1\right)}$$

$$= \left| \frac{\frac{6}{40}\,\frac{\sqrt{\omega^2+4}}{2}}{(\omega)\left(\frac{\sqrt{\omega^2+16}}{4}\right)\left(\frac{\sqrt{\omega^2+100}}{10}\right)} \right| \angle \tan^{-1}\frac{\omega}{2} - \tan^{-1}\infty - \tan^{-1}\frac{\omega}{4} - \tan^{-1}\frac{\omega}{10}$$





Ex3: Find the T.F of the following Bode plot:

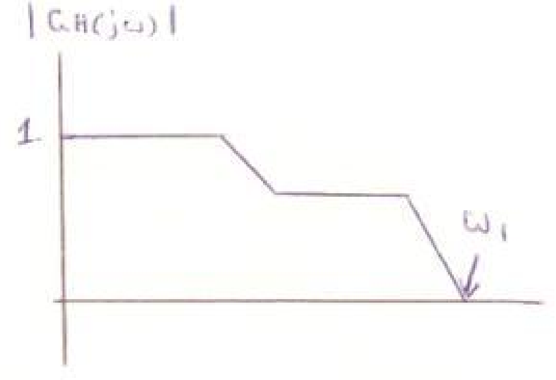


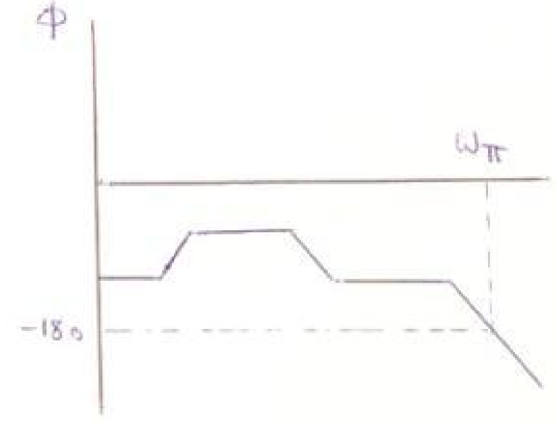
Solution:
$$GH(s) = \frac{10\left(\frac{s}{10}+1\right)}{s\,(s+1)}$$

Stability according to Bode plot

In Bode the stability is measured by phase and gain margin.

ω_1 = is the freq. when the amplitude $|GH(j\omega)| = 1$.
ω_π = is the freq. at which the phase $\phi = -180°$





Gain Margin

It is the amount of gain in decibles this can be allowed to increase in the Loop before the closed loop system reach stability.

The point of intersection of Bode phase with the Line ($\pi = -180^\circ$) determines ω_π, expand the Line to the magnitude plot, then the point of the Line with Bode determine gain margin.

The distance between this point of intersection and the real axis is equal to the gain margin in decible.

If this value is under the real axis, then the gain of the system $|GH(j\omega)| < 1$ (positive gain margin) and the system is stable.

If the value is over the real axis, then the gain margin is called negative gain margin and the system is unstable.

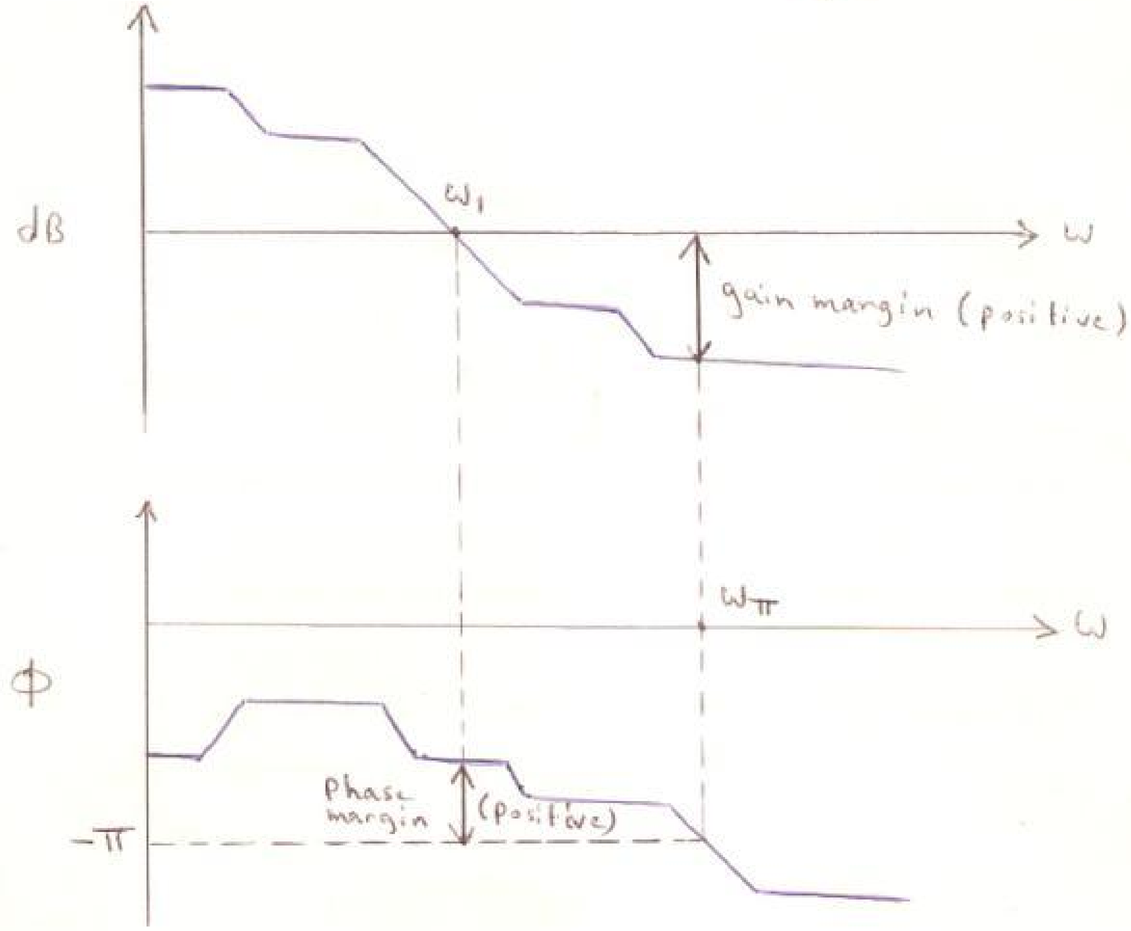
Phase Margin

The point of intersection of bode magnitude with the real axis gives ω_1, expand the Line to the phase plot, then the point of intersection of that Line with Bode phase determine the phase margin.

The difference between this point and $\phi = -180^\circ$ equal the value of the phase margin.

If this value is over the Line $\phi = -180^\circ$, then the phase margin is positive and the system is stable.

If this value is under the Line $\phi = -180^\circ$, then the phase margin is negative gain margin and the system is unstable.



$\omega_\pi > \omega_1$ stable system

$\omega_\pi = \omega_1$ critical system

$\omega_\pi < \omega_1$ unstable system

Control systems Design by the Root-Locus Method

The design by the root-Locus method is based on reshaping the root-Locus of the system by adding poles and zeros to the system's open Loop transfer function and forcing the root Loci to pass through desired closed-loop poles in the s-plane. The characteristic of the root-Locus design is its being based on the assumption that the closed-Loop system has a pair of dominant closed-Loop poles. (Zeros and additional poles affect the response characteristics).

The increasing in the gain value will improve the steady-state behavior but will result in poor stability or even instability. It is then necessary to redesign the system (by modifying the structure or by incorporating additional devices or components). Such a redesign or addition of a suitable device is called compensation. A device inserted into the system for the purpose of satisfying the specifications is called a compensator.

The compensator can be divided into:

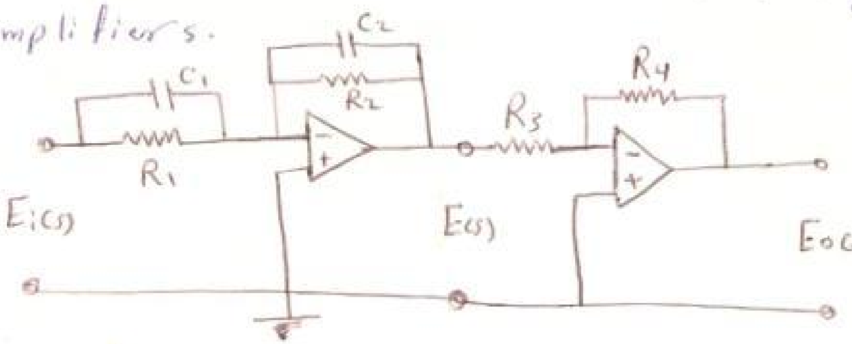
1- Lead compensator.
2- Lag compensator.
3- Lead-Lag compensator.

While the compensation can be divided into.

1- Series compensation.
2- parallel compensation or feedback compensation.

Lead Compensation

Figure below shows an electronic circuit using operational amplifiers.



The Transfer function for this circuit is :

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\,\frac{R_1C_1S+1}{R_2C_2S+1} = \frac{R_4C_1}{R_3C_2}\,\frac{S+\frac{1}{R_1C_1}}{S+\frac{1}{R_2C_2}}$$

$$= K_c\,\alpha\,\frac{TS+1}{\alpha TS+1} = K_c\,\frac{S+\frac{1}{T}}{S+\frac{1}{\alpha T}}$$

where $T = R_1C_1$; $\alpha T = R_2C_2$; $K_c = \frac{R_4C_1}{R_3C_2}$

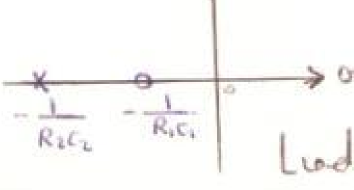
Notice that

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\,\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3} \quad ; \quad \alpha = \frac{R_2C_2}{R_1C_1}$$

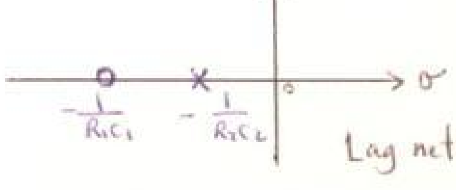
This network has a dc gain of $K_c\alpha = \frac{R_2R_4}{R_1R_3}$

The network is a Lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$.

It is a Lag network if $R_1C_1 < R_2C_2$.



Lead network



Lag network

* Lead compensation techniques based on the root-Locus approach

$$G_c(s) = K_c\alpha \frac{Ts+1}{\alpha Ts+1}$$

$$= K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \quad (0<\alpha<1)$$

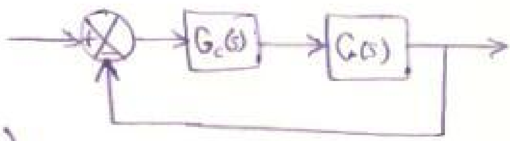
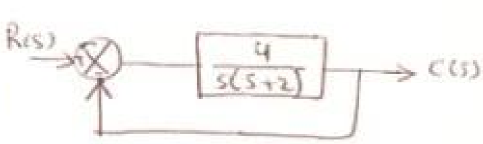


Fig. Control system

<u>Ex:</u> Consider the system shown in figure. The feed forward transfer function is

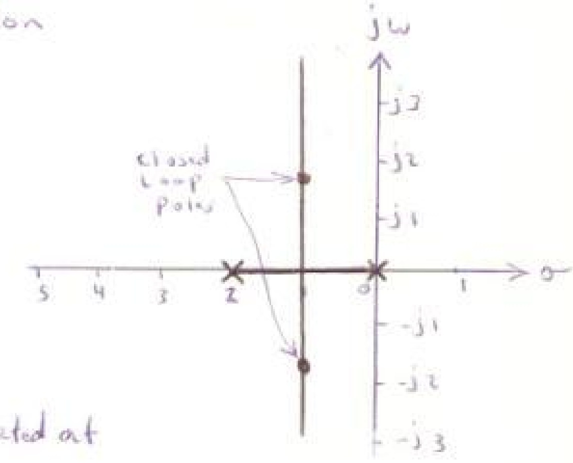


$$G(s) = \frac{4}{s(s+2)}$$

The closed Loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$$

$$= \frac{4}{(s+1+j\sqrt{3})(1+1j\sqrt{3})}$$



The closed Loop poles are Located at

$$s = -1 \mp j\sqrt{3}$$

The damping ratio $(\zeta) = 0.5$

The undamped natural frequency $(\omega_n) = 2$ rad/sec

The static velocity error constant $(K_v) = 2\ sec^{-1}$

The desired undamped natural frequency $(\omega_n) = 4$ rad/sec without changing the value of the damping ration, $\zeta = 0.5$

The desired locations of the closed Loop poles are

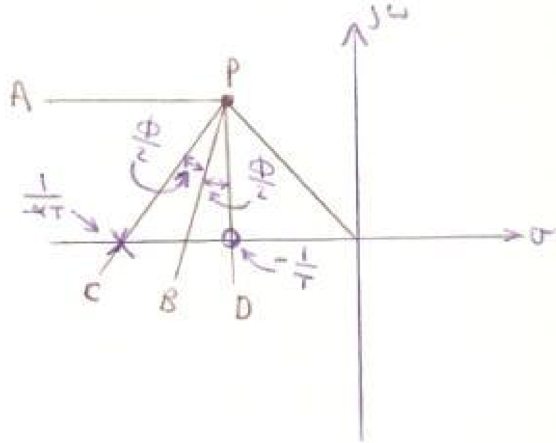
$$s = -2 \mp j2\sqrt{3}$$

The compensated system will have the open-Loop transfer function:

$$G_c(s)\,G(s) = \left(K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \right) G(s)$$

The angle of $G(s)$ at the desired closed-Loop pole is

$$\angle \frac{4}{s(s+2)} \Big|_{s=-2+j2\sqrt{3}} = -210^\circ$$



The Lead Compensator must contribute $\phi = 30^\circ$

Zero at $s = -2.9$

pole at $s = -5.4$

or $T = \frac{1}{2.9} = 0.345$; $\alpha T = \frac{1}{5.4} = 0.185$

$\therefore \alpha = 0.537$

The open Loop T. F of the compensated system becomes

$$G_c(s)\,G(s) = K_c \frac{s+2.9}{s+5.4} \frac{4}{s(s+2)} = \frac{K(s+2.9)}{s(s+2)(s+5.4)}$$

where $K = 4K_c$

$$\left| \frac{K(s+2.9)}{s(s+2)(s+5.4)} \right|_{s=-2+j2\sqrt{3}} = 1$$

or $K = 18.7$

$$G_c(s)\,G(s) = \frac{18.7(s+2.9)}{s(s+2)(s+5.4)}$$

$$K_c = \frac{18.7}{4} = 4.68$$

$K_c\alpha = 2.51$

$$G_c(s) = 2.51 \frac{0.345s+1}{0.185s+1} = 4.68 \frac{s+2.9}{s+5.4}$$

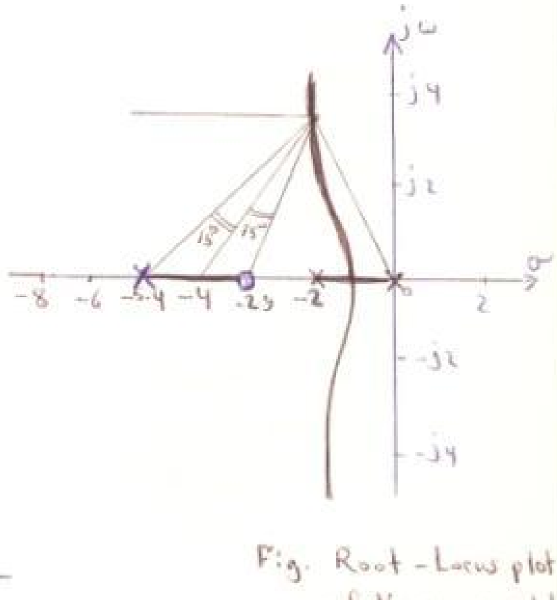


Fig. Root-Locus plot of the compensated system-

$$K_v = \lim_{s\to 0} s\,G_c(s)\,G(s) = \lim_{s\to 0} s \frac{18.7(s-2.9)}{s(s+2)(s+5.4)}$$

$$= 5.02\ \text{sec}^{-1}$$

Lag Compensation

$$\frac{E_o(s)}{E_i(s)} = K_c'\beta\,\frac{Ts+1}{\beta Ts+1} = K_c'\,\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Where $T = R_1C_1$; $\beta T = R_2C_2$; $\beta = \frac{R_2C_2}{R_1C_1} > 1$; $K_c' = \frac{R_4C_1}{R_3C_2}$

* Lag compensation techniques based on the Root - Locus approach:

$$G_c(s) = K_c'\beta\,\frac{Ts+1}{\beta Ts+1} = K_c'\,\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

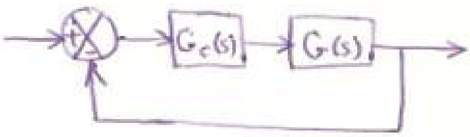


Fig. Lag Compensation

If we place the zero and pole of the Lag compensator very close to each other, then the $s = s_1$ where s_1 is one of the dominant closed - Loop poles, the magnitudes $s_1 + \frac{1}{T}$ and $s_1 + \frac{1}{\beta T}$ are almost equal, or

$$|G_c(s_1)| = \left| K_c'\,\frac{s_1+\frac{1}{T}}{s_1+\frac{1}{\beta T}} \right| \doteq K_c'$$

To make the angle contribution of the Lag portion of the compensator to be small, we require

$$-5^\circ < \angle \frac{s_1+\frac{1}{T}}{s_1+\frac{1}{\beta T}} < 0^\circ$$

The static velocity error constant (K_v) of the uncompensated system is

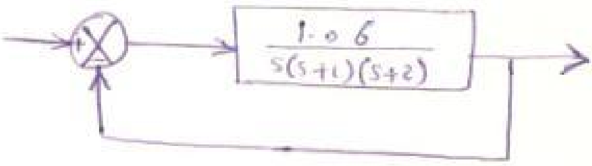
$$K_v = \lim_{s\to 0} s\,G(s)$$

For compensated system is : $K_v' = \lim_{s\to 0} s\,G_c(s)\,G(s)$

$$= \lim_{s\to 0} G_c(s)\,K_v$$

$$= K_c'\beta K_v$$

Ex: Consider the system shown in figure.



The feed forward T.F is

$$G(s) = \frac{1.06}{s(s+1)(s+2)}$$

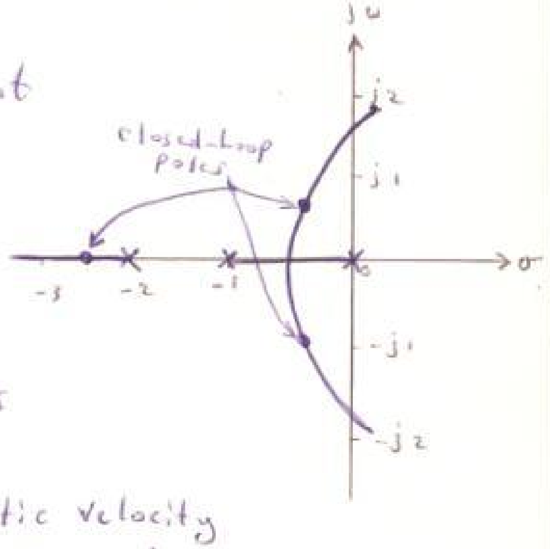
The closed-Loop T.F becomes

$$\frac{C(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

$$= \frac{1.06}{(s+0.3307-j0.5864)(s+0.3307+j0.5864)(s+2.3386)}$$

The dominant closed-Loop poles are

$$s = -0.3307 \mp j0.5864$$



The damping ratio of the dominant closed-Loop poles $\zeta = 0.491$.

The undamped natural frequency of the dominant closed-loop poles is 0.673 rad/sec.

The static velocity error constant is $0.53 \ sec^{-1}$.

It is desired to increase the static velocity error constant K_v to about $5 \ sec^{-1}$ without appreciably changing the location of the dominant closed-Loop poles.

Let us insert a lag compensator in cascade with the given feedforward transfer function.

Let us choose $\beta = 10$.

The zero and pole of the Lag compensator at $s = -0.05$ and -0.005, respectively.

$$G_c(s) = K_c' \frac{s+0.05}{s+0.005}$$

The angle contribution of this Lag network near a dominant closed-Loop poles is about 4°.

The open-Loop T.F of the compensated system then becomes

$$G_c(s)\,G(s) = K_c' \frac{s+0.05}{s+0.005} \frac{1.06}{s(s+1)(s+2)}$$

$$= \frac{K(s+0.05)}{s(s+0.005)(s+1)(s+2)}$$

where $K = 1.06\,K_c'$

If the damping ratio of the new dominant closed-Loop poles is kept the same, then the poles are obtained from the new root-Locus plot as follows:

$$s_1 = -0.31 + j0.55 \;;\; s_2 = -0.31 - j0.55$$

The open-Loop gain K is

$$K = \left| \frac{s(s+0.005)(s+1)(s+2)}{s+0.05} \right|_{s=-0.31+j0.55}$$

$$= 1.0235$$

Then the Lag compensator gain K_c' is determined as

$$K_c' = \frac{K}{1.06} = \frac{1.0235}{1.06} = 0.9656$$

Thus the T.F of the Lag compensator designed is

$$G_c(s) = 0.9656 \frac{s+0.05}{s+0.005} = 9.656 \frac{20s+1}{200s+1}$$

Then the compensated system has the following open-Loop T.F:

$$G_1(s) = \frac{1.0235(s+0.05)}{s(s+0.005)(s+1)(s+2)} = \frac{5.12(20s+1)}{s(200s+1)(s+1)(0.5s+1)}$$

The static velocity error constant K_v is

$$K_v = \lim_{s\to 0} s\,G_1(s) = 5.12 \text{ sec}^{-1}.$$

Lag - Lead Compensation

Lead compensation basically speeds up the response and increases the stability of the system. Lag Compensation improves the steady-state accuracy of the system, but reduces the speed of the response. If improvements in both transient response and steady-state response are desired, then both a Lead compensator and Lag compensator may be used simultaneously. Lag - Lead compensation combines the advantages of Lag and Lead Compensations.

Lag - Lead Compensation Techniques Based on the Root-Locus Approach.

Consider the system shown in figure.

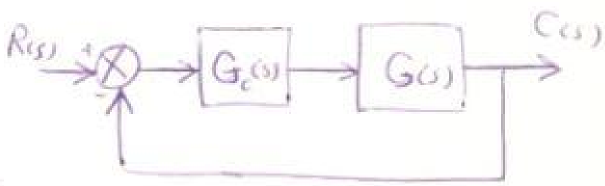


Fig. Lag - Lead Compensation

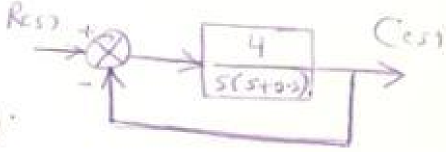
$$G_c(s) = K_c \frac{\beta}{\gamma} \frac{(T_1 s+1)(T_2 s+1)}{\left(\frac{T_1}{\gamma} s+1\right)(\beta T_2 s+1)}$$

$$= K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)$$

where $\beta > 1$ and $\gamma > 1$.

In designing Lag - Lead compensators, we consider two cases where $\gamma \neq \beta$ and $\gamma = \beta$.

Ex1: (When $\beta \neq \gamma$)



Consider the control system shown in fig. The feedforward transfer function is

$$G(s) = \frac{4}{s(s+0.5)}$$

This system has closed-Loop poles at $s = -0.25 \mp j1.9843$. The damping ratio is 0.125, the undamped natural frequency 2 rad/sec, and the static velocity error constant is $8 \ sec^{-1}$.
It is desired to make the damping ratio of the dominant closed Loop poles equal to 0.5 and to increase the undamped natural frequency to 5 rad/sec and the static velocity error constant to $80 \ sec^{-1}$.

Solution: We use a Lag-Lead compensator having the T.F

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right) \qquad (\gamma > 1 \ , \ \beta > 1) \quad (\gamma \neq \beta)$$

$$G_c(s)\, G(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right) G(s)$$

The dominant closed-Loop poles must be at

$$s = -2.5 \pm j4.33$$

Since

$$\left. \angle \frac{4}{s(s+0.5)} \right|_{s=-2.5+j4.33} = -235^\circ$$

The phase Lead portion of the Lag-Lead compensator must contribute 55° so that the root Locus passes through the desired Location of the dominant closed-Loop poles.

Choose the zero at $s = -0.5$ so that this zero will cancel the pole at $s = -0.5$ of the plant.

The pole can be Located such that the angle contribution is 55°. The pole must be Located at $s = -5.021$.

The phase Lead portion of the Lag-Lead compensator becomes

$$K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} = K_c \frac{s+0.5}{s+5.021}$$

Thus
$$T_1 = 2 \quad ; \quad \gamma = \frac{5.021}{0.5} = 10.04$$

We determine the value of K_c from the magnitude condition:

$$\left| K_c \frac{s+0.5}{s+5.021} \frac{4}{s(s+0.5)} \right|_{s=-2.5+j4.33} = 1$$

Hence
$$K_c = \left| \frac{s(s+5.021)}{4} \right|_{s=-2.5+j4.33} = 6.26$$

The phase Lag portion of the compensator can be designed as follows:

$$K_v = \lim_{s \to 0} s\, G_c(s)\, G(s) = \lim_{s \to 0} s\, K_c \frac{\beta}{\gamma} G(s)$$

$$= \lim_{s \to 0} (6.26) \frac{\beta}{10.04} \frac{4}{s(s+0.5)} = 4.988\beta = 80$$

Hence
$$\beta = \frac{80}{4.988} = 16.04$$

Finally, we choose the value of T_2 Large enough so that

$$\left| \frac{s+\frac{1}{T_2}}{s+\frac{1}{16.04T_2}} \right|_{s=-2.5+j4.33} \doteq 1$$

and

$$-5^\circ < \angle \frac{s+\frac{1}{T_2}}{s+\frac{1}{16.04T_2}} \Bigg|_{s=-2.5+j4.33} < 0^\circ$$

since $T_2 \doteq 5$ (or any number greater than 5)

We may choose $T_2 = 5$

Now
$$G_c(s) = (6.26)\left(\frac{s+\frac{1}{2}}{s+\frac{10.04}{2}} \right)\left(\frac{s+\frac{1}{5}}{s+\frac{1}{16.04 \times 5}} \right)$$

$$= 6.26\left(\frac{s+0.5}{s+5.02} \right)\left(\frac{s+0.2}{s+0.01247} \right)$$

$$= \frac{10(2s+1)(5s+1)}{(0.1992s+1)(19s+1)}$$

$$\therefore G_c(s)\, G(s) = \frac{25.04(s+0.2)}{s(s+5.02)(s+0.01247)}$$

Ex2: (When $\beta = \gamma$)

Consider the control system in previous example.

$$G_c(s) = K_c \frac{(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s+\frac{\beta}{T_1})(s+\frac{1}{\beta T_2})} \qquad (\beta > 1)$$

The desired Locations for the dominant closed-Loop poles are at

$$s = -2.5 \mp j4.33$$

The open-Loop transfer function of the compensated system is

$$G_c(s)\,G(s) = K_c \frac{(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s+\frac{\beta}{T_1})(s+\frac{1}{\beta T_2})} \cdot \frac{4}{s(s+0.5)}$$

$$K_v = \lim_{s\to 0} s\,G_c(s)\,G(s) = \lim_{s \to 0} K_c \frac{4}{0.5} = 8K_c = 80$$

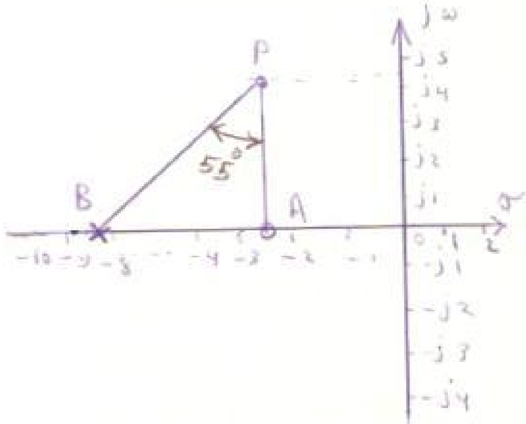
Thus $K_c = 10$

The time constant T_1 and the value of β are determined from

$$\left| \frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} \right| \left| \frac{40}{s(s+0.5)} \right|_{s=-2.5+j4.33} = 1$$

$$= \left| \frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} \right| \frac{8}{4.77} = 1$$

$$\angle \left. \frac{s+\frac{1}{T_1}}{s+\beta/T_1} \right|_{s=-2.5+j4.33} = 55^\circ$$



$\angle APB = 55^\circ$

From graphical

$$\overline{OA} = 2.38 = \frac{1}{T_1} \Rightarrow T_1 = \frac{1}{2.38} = 0.42$$

$$\overline{OB} = 8.34 = \frac{\beta}{T_1} \Rightarrow \beta = 8.34\,T_1 = 3.503$$

The phase Lead portion of the Lag-Lead network thus becomes

$$10\left(\frac{s+2.38}{s+8.34}\right)$$

From the phase Lag portion, we may choose

$$T_2 = 10$$

Then $\dfrac{1}{\beta T_2} = \dfrac{1}{3.503 \times 10} = 0.0285$

Thus, the Lag-Lead compensator becomes becomes

$$G_c(s) = (10)\left(\frac{s+2.38}{s+8.34}\right)\left(\frac{s+0.1}{s+0.0285}\right)$$

The compensated system will have the open-Loop transfer function

$$G_c(s)\,G(s) = \frac{40(s+2.38)(s+0.1)}{(s+8.34)(s+0.0285)(s+0.5)s}$$